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DELAYED DIFFERENTIAL EQUATIONS SOLVING IN A FUZZY ENVIRONMENT WITH AN ITERATIVE METHOD

[1]T. Siva Nageswara Rao, [2]N. Srinivasa Rao, [3]Ch. Subba Reddy,
[4]Renuka Kolandasamy &[5*]T.L. Yookesh

[1]Vallurupalli Nageswara Rao Vignana Jyothi Institute of Engineering &Technology (Autonomous), Hyderabad, Telangana, India – 500090.
[2]Department of Basic Science and Humanities, GMR Institute of Technology, Rajam Vizianagaram, Andhra Pradesh, India 532127
[3,5]Department of Mathematics,Vignan's Foundation for Science, Technology & Research, Guntur, Andhra Pradesh, India – 522213
[4]Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur,Andhra Pradesh, India - 522302

E.Mail: *[1]sivanageswararao_t@vnrvjiet.in, [2]srinivasarao.n@ gmrit.edu.in, [3]chreddy12@gmail.com, [4]math.renuka@kluniversity.in&[5]*renu_yookesh@yahoo.co.in*

ABSTARCT
The delay differential equation describes the different timing with different conditions in the solution. The convergence of the iterative method is studied here under various fuzzy numbers, in order to speak about its closeness and remove vagueness in the solutions. To solve the Fuzzy Delay Differential Equations(FDDE), the Runge-Kutta-Fehlberg method (RKF) is used. The numerical results are examined to demonstrate the effectiveness of the trapezoidal fuzzy number.

Key Words: Fuzzy numerical analysis, time delay, Iterative technique.

I. INTRODUCTION

A greater amount of applied design study has been done on the concept of timing under various contexts. Therefore, it is crucial to develop digital frameworks and models that can be applied to resolve often occurring doubtful situations. Chang and Zadeh were the ones who came up with the concept of a fuzzy derivative [7, 24]. The fuzzy derivative concept was then presented by Dubois and Prade [8] in respect to the augmentation rule. Kandel and Byat [13] introduced the term "fuzzy differential equation." In order to address the fuzzy differential conditions with respect to the Seikkala derivative [21], Abbasbandi and Allahviranloo [2] used mathematics. By using the variational iteration method, Jafari et al. [10] are unable to meet the nth condition of the fuzzy differential conditions. The mathematical configuration of the differential equation with fuzzy was tracked by Allahviranloo et al. [1, 3] using the appropriate indicator approach. In his book on general differential conditions and differential delay conditions, Driver [9] provides comprehensive explanations of the conditions. Bellen et al[5, 6] mathematical solutions to DDE were offered. The concepts of differential requirements for fuzzy change under summed differentiability have been addressed by Khastan et al. in [14]. Barzinji et al [4] analysis the security of a coherent state concentrated on the linear differential differential frames.

In [15,20], Renuka et.al., discussed about the concept of complementary connected domination in graphs in which the authors exhibted the results based upon cubic graphs.

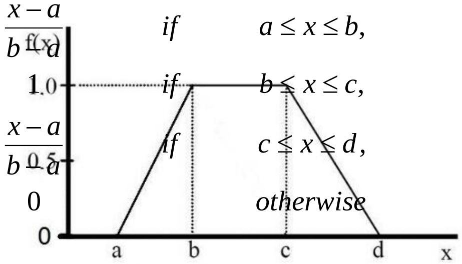
For the fourth survey, Abbasbandi and Allahwiranloo [2] presented a numerical method employing the Runge-Kutta methodology to modify fuzzy differential conditions. The Runge-Kutta approach was utilised by Pederson and Sambandham [19] to assign numerical values to fuzzy differential delay situations. Using the fourth-order Runge-Kutta strategy, Al-Rawi et al. [23] developed a numerical method for resolving the differential carry requirements. Runge-Kutta-Nystrom and the fifth Runge-Kutta technique were dropped by K. Kanagarajan et al. [12, 11] for handling fuzzy-delay differential equations. The 2nd-Runge-Kutta technique was utilised by V. Parimala et al. [18] to solve fuzzy differential situations with fuzzy initial circumstances. Using Seikkal derivative [21], Narayanamoorthy et al. [16, 17] applied Runge-third Kutta's approach to resolve fuzzy differential conditions.

The purpose of this article, which is an expansion of [16], is to examine the performance of the time lag differential condition approach in a fuzzy environment. The targeted strategy integrates seamlessly into the physical development's structure. We provide a mathematical explanation to help you comprehend the suggested tactic.

II. Preliminaries

Definition 1. The trapezoidal fuzzy number is defined by four real numbers "a, b, c, d" . A trapezoidal fuzzy number will be denoted by the membership function is defined as the

$$\mu(x) = \begin{cases} \frac{x-a}{b-a} & if \quad a \le x \le b, \\ 1 & if \quad b \le x \le c, \\ \frac{x-a}{b-a} & if \quad c \le x \le d, \\ 0 & otherwise \end{cases}$$



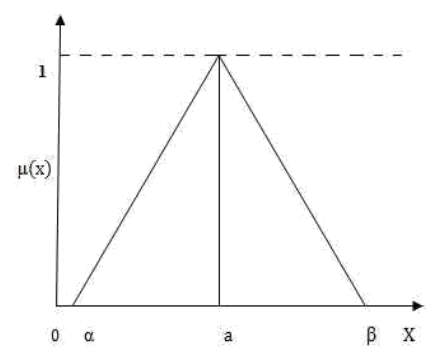


Figure 2.1: Triangular Fuzzy Number

Figure 2.1: Trapezoidal Fuzzy Number

Definition 2. Fuzzy set "$\bar{A}$" is the triangular fuzzy number with peak (or center) "a", left width $\alpha > 0$ and right $\beta > 0$, has the following form

$$\mu(x) = \begin{cases} 1-\dfrac{a-x}{\alpha} & if \quad a-\alpha < x < a, \\ 1-\dfrac{x-a}{\beta} & if \quad a < x < a+\beta, \\ 0 & otherwise. \end{cases}$$

Definition 3. For arbitraryfuzzynumbers $\psi = \left(\bar{\psi}, \underline{\psi}\right)$, $\vartheta = \left(\bar{\vartheta}, \underline{\vartheta}\right)$ the quality

$D\left(\psi, \vartheta\right) = \left\{ \sup_{0 \le \gamma \le 1} \left|\underline{\psi}(\gamma) - \underline{\vartheta}(\gamma)\right|, \sup_{0 \le \gamma \le 1} \left|\bar{\psi}(\gamma) - \bar{\vartheta}(\gamma)\right| \right\}$ is the distance between ψ *and* ϑ.

Theorem 4.

Let "p" satisfy $\left|p(d,\underline{v}) - p(d,\bar{v})\right| \le h\left(d, \left|v - \bar{v}\right|\right)$, $d \ge 0, \underline{v}, \bar{v} \in \Re$ where $h : \Re \to \Re$ will be a continuous mapping, so that $\gamma \to h(d,\gamma)$ is increasing and the initial value problem $u'(d) = h(d, u(d), u(d-\tau))$, $u(0)=0$. Has the result on $\Re$ for $u_0>0$ and that $u(d) = 0$ is the only way out of (3.1). Thus it has unique solution.

Lemma 5. Let the numbers in order $\left\{\omega_i\right\}_{i=0}^{I}$ satisfy $\left|\omega_{i+1}\right| \le G\left|\omega_i\right| + H, 0 \le i \le I-1,$

for some given constants G and H. Then $\left|\omega_i\right| \le G^i \left|\omega_0\right| + H \dfrac{G^{i-1}}{G-1}, 0 \le i \le I.$

Lemma 6. Let the arrangement of numbers $\left\{\omega_i\right\}_{i=0}^{I}$ and $\left\{\upsilon_i\right\}_{i=0}^{I}$ satisfy

$$\left|\omega_{i+1}\right| \le \left|\omega_i\right| + G \max\{\left|\omega_i\right|, \left|\upsilon_i\right|\} + H,$$
$$\left|\upsilon_{i+1}\right| \le \left|\upsilon_i\right| + G \max\{\left|\upsilon_i\right|, \left|\upsilon_i\right|\} + H$$

for some given positive constants G and H, then $V_i = \left|\omega_i\right| + \left|\upsilon_i\right|$ where $\overline{G} = 1 + 2G$ and $\overline{H} = 2H$.

III. RUNGE-KUTTA-FEHLBERG METHOD

Fuzzy delay differential equations are solved using a Runge-Kutta-Fehlberg (RKF) method that is built from scratch in this session's fuzzy environment. Let "$Q = \left[Q_1, Q_2\right]$" be the exact answer to the ambiguous beginning value problem, and "$q = \left[q_1, q_2\right]$" be the approximate solution. The solution is the calculated phase points

$$\delta = \frac{D - d_0}{N},\ d_1 = d_0 + i\delta,\ 0 \le i \le N$$

Then we obtain the,

$$\underline{q}(\kappa + 1) = \min\left(q_\kappa + \frac{25}{216}\kappa_1 + \frac{1408}{2565}\kappa_3 + \frac{2197}{4104}\kappa_4 - \frac{1}{5}\kappa_5\right);$$
$$\overline{q}(\kappa + 1) = \max\left(q_\kappa + \frac{25}{216}\kappa_1 + \frac{1408}{2565}\kappa_3 + \frac{2197}{4104}\kappa_4 - \frac{1}{5}\kappa_5\right)$$

where

$$\kappa_1(d,\gamma) = \min \delta\left(f(d_K, q_K)\right);\ \max \delta\left(f(d_K, q_K)\right)$$

$$\kappa_2(d,\gamma)=\min \delta\left(f\left(d_K+\frac{1}{4}\delta, q_K+\frac{1}{4}\kappa_1 \right)\right);\ \max \delta\left(f\left(d_K+\frac{1}{4}\delta, q_K+\frac{1}{4}\kappa_1 \right)\right)$$

$$\kappa_3(d,\gamma)=\min \delta\left(f\left(d_K+\frac{3}{8}\delta, q_K+\frac{3}{32}\kappa_1+\frac{9}{32}\kappa_2 \right)\right);\ \max \delta\left(f\left(d_K+\frac{3}{8}\delta, q_K+\frac{3}{32}\kappa_1+\frac{9}{32}\kappa_2 \right)\right)$$

$$\kappa_4(d,\gamma)=\min \delta\left(f\left(d_K+\frac{12}{13}\delta, q_K+\frac{1932}{2197}\kappa_1-\frac{7200}{2197}\kappa_2+\frac{72916}{2197}\kappa_3 \right)\right);$$

$$\max \delta\left(f\left(d_K+\frac{12}{13}\delta, q_K+\frac{1932}{2197}\kappa_1-\frac{7200}{2197}\kappa_2+\frac{72916}{2197}\kappa_3 \right)\right)$$

$$\kappa_5(d,\gamma)=\min \delta\left(f\left(d_K+\delta, q_K+\frac{439}{216}\kappa_1-8\kappa_2+\frac{3680}{513}\kappa_3-\frac{845}{4104}\kappa_4 \right)\right);$$

$$\max \delta\left(f\left(d_K+\delta, q_K+\frac{439}{216}\kappa_1-8\kappa_2+\frac{3680}{513}\kappa_3-\frac{845}{4104}\kappa_4 \right)\right)$$

$$\kappa_6(d,\gamma)=\min \delta\left(f\left(d_K+\frac{1}{2}\delta, q_K-\frac{8}{27}\kappa_1+2\kappa_2-\frac{3544}{2565}\kappa_3+\frac{1859}{4104}\kappa_4-\frac{11}{40}\kappa_5 \right)\right);$$

$$\max \delta\left(f\left(d_K+\frac{1}{2}\delta, q_K-\frac{8}{27}\kappa_1+2\kappa_2-\frac{3544}{2565}\kappa_3+\frac{1859}{4104}\kappa_4-\frac{11}{40}\kappa_5 \right)\right)$$

Observance argument the value of γ is fixed and then the exact and approximate solution of d_η are represented by,

$$[Q(d_\eta)]_\gamma = [Q_1(d_\eta;\gamma), Q_2(d_\eta;\gamma)]$$
$$[q(d_\eta)]_\gamma = [q_1(d_\eta;\gamma), q_2(d_\eta;\gamma)],\ 0 \le \eta \le N \tag{3.1}$$

By this, we finding $[q_1(d_\eta;\gamma), q_2(d_\eta;\gamma)]$ converges to $[Q_1(d_\eta;\gamma), Q_2(d_\eta;\gamma)]$ respectively at $\delta \to 0$.

Theorem 3.1.

Consider the systems (3.1), for $\gamma \in [0,1]$, then $\lim\limits_{f\to 0}\left[\underline{q}(d_\eta)\right]_\gamma = \underline{Q}\left(d_\eta;\gamma\right); \lim\limits_{f\to 0}\left[\overline{q}(d_\eta)\right]_\gamma = \overline{Q}\left(d_\eta;\gamma\right)$.

IV. ILLUSTRATIVE EXAMPLE

Consider the linear fuzzy delay differential equation

$$\varphi'(i) = \frac{1}{2}e^{\frac{i}{2}}\varphi\left(\frac{i}{2}\right) + \frac{1}{2}\varphi(i) \tag{4.1}$$

subject to initial conditions $\varphi(0)=1$, then by Definition 1, we have $\varphi(0)=(0.8+0.125\alpha,1.1-0.1\alpha)$

The exact solution of (4.1) is

$$\underline{\varphi}(i,\alpha)=(0.8+0.125\alpha)e^{i}$$
$$\overline{\varphi}(i,\alpha)=(1.1-0.1\alpha)e^{i} \quad (4.2)$$

To approximate (4.1) by the RKF method, it can be written in an operator form

$$\underline{\varphi}'(i,\alpha)=\frac{1}{2}e^{\frac{i}{2}}\underline{\varphi}\left(\frac{i}{2}\right)+\frac{1}{2}\underline{\varphi}(i)$$
$$\overline{\varphi}'(i,\alpha)=\frac{1}{2}e^{\frac{i}{2}}\overline{\varphi}\left(\frac{i}{2}\right)+\frac{1}{2}\overline{\varphi}(i) \quad (4.3)$$

With the initial condition $\underline{\varphi}(0)=0.8+0.125\alpha$; $\overline{\varphi}(0)=1.1-0.1\alpha$

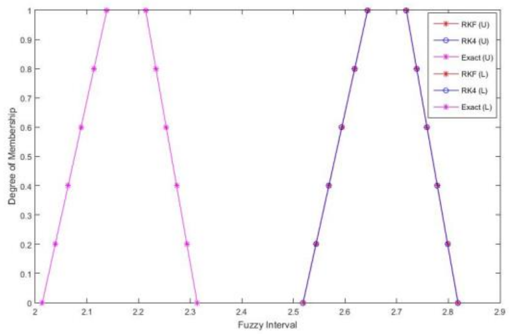
Table 4.1: Approximate Values

	Runge-Kutta Fehlberg method		Runge-Kutta fourth order method	
α	$\underline{\varphi}(i)$	$\overline{\varphi}(i)$	$\underline{\varphi}(i)$	$\overline{\varphi}(i)$
0	1.718802	3.718802	1.718282782	3.718282782
0.2	1.918802	3.518802	1.918282782	3.518282782
0.4	2.118802	3.318802	2.118282782	3.318282782
0.6	2.318802	3.118802	2.318282782	3.118282782
0.8	2.518802	2.918802	2.518282782	2.918282782
1	2.718802	2.718802	2.718282782	2.718282782

Table 4.2: Error Table

	Exact Solution		**Runge-Kutta Fehlberg method**		**Runge-Kutta fourth order method**	
γ	$\underline{U}(\chi,\gamma)$	$\overline{U}(\chi,\gamma)$	$\underline{u}(\chi,\gamma)$	$\overline{u}(\chi,\gamma)$	$\underline{u}(\chi,\gamma)$	$\overline{u}(\chi,\gamma)$
0	0	5.436563657	1.719012514	1.71755143	1.71828278	1.718280875
0.2	0.543656366	4.892907291	1.375356148	1.37389478	1.37462642	1.374624509
0.4	1.087312731	4.349250925	1.031699783	1.03023841	1.03097005	1.030968143
0.6	1.630969097	3.805594559	0.680932157	0.68658205	0.68731369	0.687311777
0.8	2.174625463	3.261938194	0.337275788	0.34292568	0.34365732	0.343655412
1	2.718281828	2.718281828	0.000730686	0.00073069	$9.54*10^{-7}$	$9.54*10^{-7}$

Examination of above results are compared in following figures, in figure 4.1 both the method are compared with the exact solution and in figure 4.2 shows the bond between the two method.



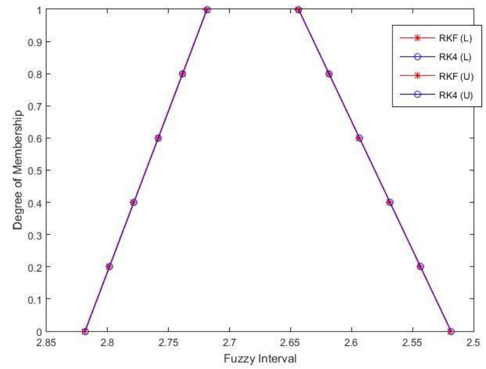


Figure 4.1: Comparison Between Approximate and Exact Solution Figure 4.2: Comparision Between RKF and RK4th order Methods

On with Triangular Fuzzy number (Definition 2), $u(0) = (\gamma, 2-\gamma)$

The exact solution of (4.1) is

$$\underline{u}(\chi,\gamma) = \gamma e^{\chi}$$
$$\overline{u}(\chi,\gamma) = (2-\gamma)e^{\chi} \tag{5.2}$$

To approximate (5.1) by the RKF method, and general form as (5.3)

$$\underline{u}'(\chi,\gamma) = \frac{1}{2}e^{\frac{\chi}{2}}\underline{u}\left(\frac{\chi}{2}\right) + \frac{1}{2}\underline{u}(\chi)$$
$$\overline{u}'(\chi,\gamma) = \frac{1}{2}e^{\frac{x}{2}}\overline{u}\left(\frac{\chi}{2}\right) + \frac{1}{2}\overline{u}(\chi) \tag{5.3}$$

with the initial condition $\underline{u}(0) = \gamma,\ \overline{u}(0) = 2-\gamma.$

The value of the approximate method is tabulated below with $\chi = 1.$

Table 4.3: Approximate Values

	Runge-Kutta Fehlberg method		Runge-Kutta fourth order method	
γ	$\underline{u}(\chi,\gamma)$	$\overline{u}(\chi,\gamma)$	$\underline{u}(\chi,\gamma)$	$\overline{u}(\chi,\gamma)$
0	1.718802	3.718802	1.718282782	3.718282782
0.2	1.918802	3.518802	1.918282782	3.518282782

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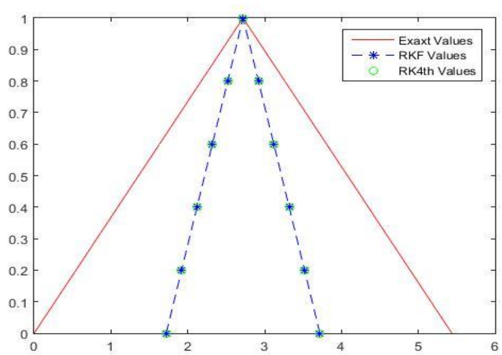
0.4	2.118802	3.318802	2.118282782	3.318282782
0.6	2.318802	3.118802	2.318282782	3.118282782
0.8	2.518802	2.918802	2.518282782	2.918282782
1	2.718802	2.718802	2.718282782	2.718282782

Table 4.4: Error Table

	Exact Solution		**Runge-Kutta Fehlberg method**		**Runge-Kutta fourth order method**	
γ	$\underline{U}(\chi,\gamma)$	$\overline{U}(\chi,\gamma)$	$\underline{u}(\chi,\gamma)$	$\overline{u}(\chi,\gamma)$	$\underline{u}(\chi,\gamma)$	$\overline{u}(\chi,\gamma)$
0	0	5.436563657	1.719012514	1.71755143	1.71828278	1.718280875
0.2	0.543656366	4.892907291	1.375356148	1.37389478	1.37462642	1.374624509
0.4	1.087312731	4.349250925	1.031699783	1.03023841	1.03097005	1.030968143
0.6	1.630969097	3.805594559	0.680932157	0.68658205	0.68731369	0.687311777
0.8	2.174625463	3.261938194	0.337275788	0.34292568	0.34365732	0.343655412
1	2.718281828	2.718281828	0.000730686	0.00073069	$9.54*10^{-7}$	$9.54*10^{-7}$

Examination of above results are compared in following figures, in figure 4.3 both the method are compared with the exact solution and in figure 4.4 shows the bond between the two method.



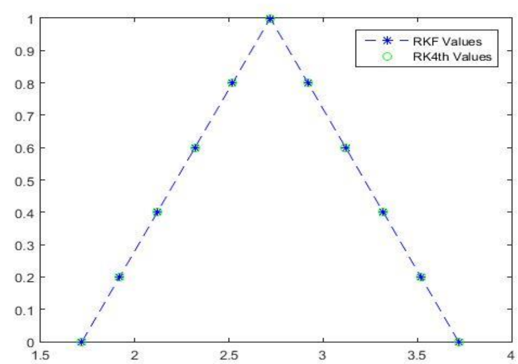


Figure 4.3: Comparison Between Approximate and Exact SolutionFigure 4.4: Comparision Between RKF and RK4th order Methods

V. Conclusion

Two distinct fuzzy numbers are used to test the efficacy of the suggested method for solving fuzzy delay difference equations. The accuracy of the suggested technique is demonstrated using a numerical example, which demonstrates the performance of the implement number system and compares it to that of the precise and Runge-Kutta Fehlberg method. When compared to its triangular counterpart, the trapezoidal fuzzy number exhibits superior convergence.

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