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Common Fixedpoint Theorem For Six Mappings In Cone Metric Spaces

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Abstract
Sessa [13], initiated the tradition of improving commutativity conditions in metrical common fixed point theorems. While doing so Sessa [13] introduced the notion of weak commutativity. Motivated by Sessa [13], Jungck [10] defined the concept of compatibility of two mappings, which includes weakly commuting mappings as a proper subclass. Jungck and Rhoades [11] introduced the notion of weakly compatible mappings, which is weaker than compatibility. In this paper, by using notions of compatibility, weak compatibility and commutativity, we prove some common fixed point theorems for six mappings involving rational contractive condition in complete cone metric spaces. Our work generalizes some earlier results of Goyal([3], [4]),Jeong-Rhoades [8], and others.

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1 INTRODUCTION:

Huang and Zhang [6] generalized the concept of a cone metric space, re-placing the set of real numbers by an ordered Banach space and obtained some common fixed point theorems for mappings satisfying different contractive conditions over cone metric space. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades[2] studied common fixed point theorems in cone metric spaces. Moreover, Huang and Zhang [6], Abbas and Jungck[1], IIlic and Rakocevic [7] proved their results for normal cones. Jungck [10] generalized the concept of weak commuting by defining the term compatible mappings and proved that the weakly commuting mappings are compatible but the converse is not true. Jungck and Rhoades [11] defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space utilizing these concepts. In this paper, by using notions of compatibility, weak compatibility and commutativity, we prove some common fixed point theorems for six mappings involving rational contractive condition in complete cone metric spaces. Our work generalizes some earlier results of Goyal([3], [4]),Jeong-Rhoades [3], and others.

Some examples are also furnished to demonstrate the validity of the hypothesis.

2. BASIC DEFINITIONS:

The following definitions are in literature of Huang and Zhang [6].

Definition 2.1: Let E be a real Banach space and P be a subset of E. The subset P is called a cone if and onlyif
(a) P is closed, non-empty and $P \neq \{0\}$
(b) $a, b \in R,\ a, b \geq 0,\ x, y \in P$ implies
$ax + by \in P$
(c) $x \in P$ and $-x \in P \Rightarrow x = 0$
i.e $P \cap (-P) = \{0\}$

Definition 2.2: Let P be a cone in a Banach space Ei.e. given a cone $P \subset E$, define partial ordering '$\leq$' with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$while $x \ll y$ will stand for $y - x \in Int\ P$, where $Int\ P$ denote the interior of the set P. This cone P is called an order cone.

Definition 2.3: Let E be a real Banach space and $P \subset E$ be an order cone. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,
$0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ … (1)

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 2.4: Let X be a non-empty and E be a real Banach space. Suppose that the mapping $d: X \times X \to E$satisfies
(d1) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$
(d2) $d(x,y) = d(y,x)$ for all $x,y \in X$
(d3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$

Then d is called a cone metric on X and (X,d) is called a cone metric space.
It is obvious that cone metric spaces generalize metric spaces because each metric space is a cone metric space with $E = R$ and $P = [0,+\infty[$

Example 2.5: (a) Let $E = R^2$, $P = \{(x,y) \in E \,|x,y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \to E$ such that
$d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant.

Then (X,d) is a cone metric space.
(b) Let $E = R^n$ with $P = \{(x_i, \dots, x_n): x_i \geq 0, \forall i = 1,2,\dots,n\}X = R$ and $d: X \times X \to E$ such that
$$d(x,y) = (|x-y|, \alpha_i|x-y|, \dots, \alpha_{n-1}|x-y|)$$
where $\alpha_i \geq 0$ for all $1 \leq i \leq n-1$. Then (X,d) is a cone metric space.

Definition 2.6: Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$is a

(a) convergent sequence or $\{x_n\}$ converges to x if for every c in E with $c \gg 0$, there is an $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) \ll c$ for some fixed point x in X where x is that limit of $\{x_n\}$. This is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$. Completeness is defined in the standard way.

It was proved in [5] if (X,d) be a cone metric space, P be a normal cone with normal constant K and $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(b) Cauchy sequence if for c in E with $c \gg 0$, there is an $n_0 \in N$ such that for all $n,m > n_0$, $d(x_n, x_m) \ll c$.

It was proved in [5] if (X,d) be a cone metric space, P be a normal cone with normal constant K and $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequnence if and only if $d(x_n, x_m) \to 0$ as $n,m \to \infty$.

Definition2.7: A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence in unique provided P is a normal cone with normal constant K.

In recent years several definitions of conditions weaker than commutativity have appeared which facilitated significantly to extend the Jungck's [10] theorem and several others. Foremost among of them is perhaps the weak commutativity condition introduced by Sessa [13] which can be described as follows:

Definition 2.8: Let S and T be mappings from a cone metric space (X,d) into itself. Then S and T are said to be weakly commuting mappings on X if
$d(STx, TSx) \leq d(Sx, Tx)$, for all $x \in X$.
obviously a commuting pair is weakly commuting but its converse need not be true as is evident from the following example.

Example 2.9: Consider the set $X = [0,1]$ with the usual metric defined by
$$d(x,y) = |x-y| = \|x-y\|$$

Define S and $T: X \to X$ by
$Sx = \frac{x}{3-2x}$ and $Tx = \frac{x}{3}$ for all $x \in X$.

Then, we have to any x in X
$STx = \frac{x}{9-2x}$ and $TSx = \frac{x}{9-6x}$

Hence $ST \neq TS$. Thus, S and T do not commute.
Again,
$$d(STx, TSx) = \left\|\frac{x}{9-2x} - \frac{x}{9-6x}\right\| = \frac{4x^2}{(9-2x)(9-6x)} \leq \frac{2x^2}{3(3-2x)} = \frac{x}{3-2x} - \frac{x}{3} = d(Sx, Tx)$$
and thus S and T commute weakly.

Example2.10: Consider the set $X = [0,1]$ with the usual metric $d(x,y) = \|x-y\|$. Let $Sx = \frac{x}{2}$ and $Tx = \frac{x}{2+x}$, for every $x \in X$. Then, for all $x \in X$
$STx = \frac{x}{4+2x}$ and $TSx = \frac{x}{4+x}$

Hence, $ST \neq TS$. Thus, S and T do not commute.
Again, $d(STx, TSx) = \left\|\frac{x}{4+2x} - \frac{x}{4+x}\right\|$

$$= \frac{x^2}{(4+x)(4+2x)}$$

$$\leq \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x}$$

$$= d(Sx, Tx)$$

and thus, S and T commute weakly.

Obviously, the class of weakly commuting is wider and includes commuting mappings as subclass.

Jungck [10] has observed that for $X = R$ if $Sx = x^3$ and $Tx = 2x^3$ then S and T are not weakly commuting. Thus it is desirable to a less restrictive concept which he termed as 'compatibility' the class of compatible mappings is still wider and includes weakly commuting mappings as subclass as is evident from the following definition of Jungck[10].

Definition2.11: Let S and T be self mappings on a cone metric space (X, d). Then S and T are said to be compatible mappings on X if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some point $t \in X$.

Obviously, any weakly commuting pair $\{S, T\}$ is compatible, but the converse is not necessarily true, as in the following example.

Example2.12: Let $Sx = x^3$ and $Tx = 2x^3$ with $X = R$ with the usual metric. Then S and T are compatible, since $|Tx - Sx| = |x^3| \to 0$ if and only if $|STx - TSx| = 6|x^9| \to 0$ But $|STx - TSx| \leq |Tx - Sx|$ is not true for all $x \in X$, say for example at $x = 1$.

Definition 2.13: Let S and T be self maps of a set X. If $w = Sx = Tx$ for some x in X, then x is called a coincidence point of S and T and w is called a point o coincidence of S and T.

Definition2.14: A pair of self mappings (S, T) on a cone metric space (X, d) is said to be weakly compatible if the mappings commute at their coincidence points i.e $Sx = Tx$ for some $x \in X$ implies that $STx = TSx$.

Example2.15: Let X = [2, 20] with usual metric define

$$Tx = \begin{cases} 2 & if \quad x = 2 \\ 12 + x & if \quad 2 < x \leq 5 \\ x - 3 & if \quad 5 < x \leq 20 \end{cases} \quad and \quad Sx = \begin{cases} 2 & if \quad x \in \{2\} \cup (5, 20] \\ 8 & if \quad 2 < x \leq 5 \end{cases}$$

S and T are weakly compatible mappings which is not compatible.

Remark 2.16: Let (X, d) be a cone metric space with a cone P. If $d(x, y) \leq hd(x, y)$ for all $x, y \in X, h \in (0,1)$, then $d(x, y) = 0$, which implies that $x = y$.

3MAIN RESULTS:-

LetR^+ be the set of non-negative real numbers and let$F: R^+ \to R^+$ be a mapping such that$F(0) = 0$ and F is continuous at 0.

The following Lemma is the key in proving our result. Its proof is similar to that of Jungck [9]

Lemma3.1: Let$\{y_n\}$be a sequence in a completemetric space(X, d). If there exists a $k \in (0,1)$such that

$$\|d(y_{n+1}, y_n)\| \leq k\|d(y_{n+1}, y_n)\|$$

for all$n$, then $\{y_n\}$ converges to a point in X.

Motivated by the contractive condition given by contractive conditipn of Slobodan C. Nesic [12], Jeong and Rhoades [8] we prove the following theorem.

Theorem3.2:Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose mappings $A, B, S, T, I, J: X \to X$satisfying $AB(X) \subset J(X)$, $ST(X) \subset I(X)$and for each$x, y \in X$ either,

$$d(ABx, STy) \leq \alpha_1 \left[\frac{d(ABx, Jy).d(Jy, STy) + d(STy, Ix)d(Ix, ABx)}{d(ABx, Jy) + d(STy, Ix)}\right] + \alpha_2[d(ABx, Jy) + d(Jy, STy)] + \alpha_3 d(Ix, ABx) + F\big(d(STy, Ix).d(Ix, ABx)\big) \quad ....(2)$$

If$d(ABx, Jy) + d(STy, Ix) \neq 0$, $\alpha_i \geq 0$ $(i = 1,2,3, \ldots)$with at least one α_i non zero and$\alpha_1 + 2\alpha_2 + \alpha_1 \leq 1$,

$d(ABx, STy) = 0$ if $d(ABx, Jy) + d(STy, Ix) = 0$ ... (3)
if either
(a) $(AB, I)$are compatible, I or AB is continuous and(ST, J) are weakly compatible or$(a')(ST, J)$ are compatible, J or ST is continuous thenAB, ST, I and J have a unique common fixed point. Furthermore if the pairs(A, B), (A, I), $(B, I), S, T), (S, J)$ and (T, J) are commuting mappings thenA, B, S, T, I and J have a unique common fixed point.

We construct a sequence as follows. Letx_0 be an arbitrary point in X. Since$AB(X) \subseteq J(X)$ we can choose a pointx_1 in X such that$ABx_0 = Jx_1$. Again since$ST(X) \subseteq I(X)$ we can choose a pointx_2 in X such that$STx_1 = Ix_2$, construct a sequence$\{z_n\}$be repeatedly using this argument, $x_{2n} = ABx_{2n} = Jx_{2n+1}$, $z_{2n+1} = STx_{2n+1} = Ix_{2n+2}$, $n = 0,1,2, \ldots$
Let us put $U_{2n} = d(ABx_{2n}, STx_{2n+1})$ and $U_{2n+1} = d(STx_{2n+1}, ABx_{2n+2})$ for $n = 0,1,2, \ldots$Now we distinguish two cases :

Proof :
Case – 1 :
Suppose that $U_{2n} + U_{2n+1} \neq 0$ for $n = 0,1,2, \ldots$ then on using inequality (2), we have

$$U_{2n+1} = d(z_{2n+1}, z_{2n+2}) = d(STx_{2n+1}, ABx_{2n+2})$$

$$\leq \alpha_1 \left[\frac{d(ABx_{2n+2}, Jx_{2n+1}). d(Jx_{2n+1}, STx_{2n+1}) + d(STx_{2n+1}, Ix_{2n+2}). d(Ix_{2n+2}, ABx_{2n+2})}{d(ABx_{2n+2}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n+2})}\right]$$

$$+\alpha_2[d(ABx_{2n+2}, Jx_{2n+2}) + d(Jx_{2n+1}, STx_{2n+1})] + \alpha_3 d(Ix_{2n+2}, Jx_{2n+1})$$

$$+F\left(d(STx_{2n+1}, Ix_{2n+2}). d(Ix_{2n+2}, ABx_{2n+2})\right)$$

$$\leq \alpha_1 \left[\frac{d(z_{2n+2}, z_{2n}). d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+1}). d(z_{2n}, z_{2n+2})}{d(z_{2n+2}, z_{2n}) + d(z_{2n}, z_{2n+1})}\right]$$

$$+\alpha_2[d(z_{2n+2}, z_{2n+1}) + d(z_{2n}, z_{2n+1})] + \alpha_3 d(z_{2n+1}, z_{2n})$$

$$+F\left(d(z_{2n+1}, z_{2n+1}). d(z_{2n+1}, z_{2n+2})\right)$$

$$= \alpha_1 d(z_{2n}, z_{2n+1}) + +\alpha_3 d(z_{2n+2}, z_{2n+1}) + \alpha_2 d(z_{2n}, z_{2n+1}) + \alpha_3 d(z_{2n+1}, z_{2n}) + F(0)$$

which implies that

$$d(z_{2n+1}, z_{2n+2}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3}{(1 - \alpha_2)} d(z_{2n}, z_{2n+1}) \leq kd(z_{2n}, z_{2n+1})$$

where, $t = \frac{\alpha_1+\alpha_2+\alpha_3}{(1-\alpha_2)} < 1$

Similarly we can conclude that

$$d(z_{2n}, z_{2n+1}) \leq td(z_{2n-1}, z_{2n})$$

Thus for every n, we have

$$d(z_n, z_{n+1}) \leq td(z_{n-1}, z_n) \quad \ldots (4)$$

Therefore, by (1) we have,

$$\|d(z_n, z_{n+1})\| \leq \ tK\|d(z_{n-1}, z_n)\|$$

By Lemma 3.1$\{z_n\}$ converges to some$z \in X$. Hence, the sequences$ABx_{2n} = Jx_{2n+1}$ and $STx_{2n+1} = Ix_{2n+2}$, which are subsequences also converges to the some point z.

Let us now assume that I is continuous so that the sequence$\{I^2x_{2n}\}$ and $\{IABx_{2n}\}$converges to the same pointIz. Also(AB, I) are compatible, so sequence $\{ABIx_{2n}\}$also converges to Iz.

Now,

$$d(ABIx_{2n}, STx_{2n+1}) \leq \alpha_1 \left[\frac{d(ABIx_{2n}, Jx_{2n+1}).d(Jx_{2n+1}, STx_{2n+1}) + d(STx_{2n+1}, I^2x_{2n}).d(I^2x_{2n}, ABIx_{2n})}{d(ABx_{2n}, Jx_{2n+1}) + d(STx_{2n+1}, I^2x_{2n})}\right]$$

$$+\alpha_2[d(ABIx_{2n}, I^2x_{2n}) + d(Jx_{2n+1}, STx_{2n+1})]$$

$$+\alpha_3 d(I^2x_{2n}, Jx_{2n+1})$$

$$+F\big(d(STx_{2n+1}, I^2x_{2n}).d(I^2x_{2n}, ABIx_{2n})\big)$$

which on using (5.1) and letting$n \to \infty$ reduces to

$$d(Iz, z) \leq \alpha_1 \left[\frac{d(Iz, z).d(z.z) + d(Iz, z).d(Iz, Iz)}{d(Iz, z) + d(z, Iz)}\right] + \alpha_2[d(Iz, Iz) + d(z, z)]$$

$$+\alpha_3 d(Iz, z) + F\big(d(Iz, z).d(Iz, Iz)\big)$$

or $\quad d(Iz, z) \leq \alpha_3 d(Iz, z)$

Yielding thereby$Iz = z$.
Now,

$$d(ABz, STx_{2n+1}) \leq \alpha_1 \left[\frac{d(ABz, Jx_{2n+1}).d(Jx_{2n+1}, STx_{2n+1}) + d(STx_{2n+1}, Iz).d(Iz, ABz)}{d(ABz, Jx_{2n+1}) + d(STx_{2n+1}, Iz)}\right]$$

$$+\alpha_2[d(ABz, Iz) + d(Jx_{2n+1}, STx_{2n+1})]$$

$$+\alpha_3 d(Iz, Jx_{2n+1}) + F\big(d(STx_{2n+1}, Iz).d(Iz, ABz)\big)$$

which on using (1) and letting $n \to \infty$reduces to

$$d(ABz, z) \leq \alpha_1 \left[\frac{d(ABz, z).d(z.z) + d(z, Iz).d(Iz, ABz)}{d(ABz, z) + d(z, Iz)}\right] + \alpha_2[d(ABz, Iz) + d(z, z)]$$

$$+\alpha_3 d(Iz, z) + F\big(d(z, Iz).d(Iz, ABz)\big)$$

Using$Iz = z$, we get

$$d(ABz, z) \leq \alpha_2 d(ABz, z)$$

Implying thereby$ABz = z$.
Since$AB(X) \subset J(X)$, there always exist a point z'such that$Jz' = z$ so that$STz = ST(Jz')$.
Now, using (2), we have

$$d(z, STz') = d(ABz, STz')$$

$$\leq \alpha_1 \left[\frac{d(ABz, Jz').d(Jz', STz') + d(STz', Iz).d(Iz, ABz)}{d(ABz, Jz') + d(STz', Iz)}\right] + \alpha_2[d(ABz, Iz) + d(Jz', STz')]$$

$$+\alpha_3 d(Iz, Jz') + F\big(d(STz', Iz).d(Iz, ABz)\big)$$

$$= \alpha_1 \left[\frac{d(z, z).d(z, STz') + d(STz', z).d(z, z)}{d(z, z) + d(STz', z)}\right] + \alpha_2[d(z, z) + d(z, STz')] + \alpha_3 d(z, z)$$

$$+F\big(d(STz', z).d(z, z)\big)$$

$$= \alpha_2 d(z, STz') + F(0)$$

or, $\quad d(z, STz') \leq \alpha_2 d(z, STz')$, which implies that$STz' = z = Jz'$. It shows that (*ST*,*J*) have a coincidence pointz' Now using the weak compatibility of (*ST*, *J*), we have

$$STz = ST(Jz') = J(STz') = Jz$$

which shows that z is also a coincidence point of the pair (*ST*, *J*).
Now,

$$d(z,STz) = d(ABz,STz)$$

$$\leq \alpha_1\left[\frac{d(ABz,Jz).d(Jz,STz) + d(STz,Iz).d(Iz,ABz)}{d(ABz,Jz) + d(STz,Iz)}\right]+\alpha_2[d(ABz,Iz) + d(Jz,STz)]$$

$$+\alpha_3 d(Iz,Jz) + F\big(d(ABz,Iz).d(Iz,STz)\big)$$

$$= \alpha_1\left[\frac{d(z,STz).d(STz,STz) + d(STz,z).d(z,z)}{d(z,STz) + d(STz,z)}\right]+\alpha_2[d(z,z) + d(STz,STz)]$$

$$+\alpha_3 d(z,STz) + F\big(d(z,z).d(z,STz)\big)$$

$$= \alpha_3 d(z,STz) + F(0)$$

or, $\quad d(z,STz) \leq \alpha_3 d(z,STz).$

Hence,$z = STz = Jz$, which shows that z is a common fixed point of *AB*, *I*,*ST* and *J*.
Now, we suppose that *AB* is continuous so that the sequence$\{AB^2x_{2n}\}$ and$\{ABIx_{2n}\}$ converges toABz. Since(AB, I) are compatible it follows that$\{IABx_n\}$ also converges to ABz. Thus,

$$d(AB^2x_{2n},STx_{2n+1}) \leq \alpha_1\left[\frac{d(AB^2x_{2n},Jx_{2n+1}).d(Jx_{2n+1},STx_{2n+1})+d(STx_{2n+1},IABx_{2n+1}).d(IABx_{2n},AB^2x_{2n})}{d(ABz,Jx_{2n+1})+d(STx_{2n+1},Iz)}\right]$$
$$+\alpha_2 d(AB^2x_{2n},IABx_{2n}) + d(Jx_{2n+1},STx_{2n+1})$$
$$+\alpha_3 d(IABx_{2n},Jx_{2n+1})$$
$$+F\big(d(STx_{2n+1},IABx_{2n}).d(IABx_{2n},AB^2x_{2n})\big)$$

which on using (1) and letting $n \to \infty$reduces to

$$d(ABz,z) \leq \alpha_1\left[\frac{d(ABz,z).d(z.z) + d(z,ABz).d(ABz,ABz)}{d(ABz,z) + d(z,Iz)}\right] + \alpha_2[d(ABz,ABz) + d(z,z)]$$
$$+\alpha_3 d(ABz,z) + F\big(d(z,ABz).d(ABz,ABz)\big)$$

or, $\quad d(ABz,z) \leq \alpha_3 d(ABz,z)$
yielding thereby$ABz = z$.

As earlier, there exists a pointz' in X such that$ABz = z = Jz'$. Then,

$$d(AB^2x_{2n},STz') \leq \alpha_1\left[\frac{d(AB^2x_{2n},Jz').d(Jz',STz') + d(STz',IABx_{2n}).d(IABx_{2n},AB^2x_{2n})}{d(AB^2x_{2n},Jz') + d(STz',IABx_{2n})}\right]$$
$$+\alpha_2 d(AB^2x_{2n},IABx_{2n}) + d(Jz',STz')+\alpha_3 d(IABx_{2n},Jz')$$
$$+F\big(d(SIz',IABx_{2n}).d(IABx_{2n},AB^2x_{2n})\big)$$

which on letting $n \to \infty$ reduces to

$$d(z,STz') \leq \alpha_1\left[\frac{d(ABz,Jz').d(Jz'.STz') + d(STz',ABz).d(ABz,ABz)}{d(ABz,Jz') + d(STz',ABz)}\right]$$
$$+\alpha_2[d(ABz,ABz) + d(Jz',STz')]+\alpha_3 d(ABz,Jz')$$
$$+F\big(d(STz',ABz).d(ABz,ABz)\big)$$

or, $\quad d(z,STz') \leq \alpha_2 d(z,STz')$.
This gives$STz' = z = Jz'$.

Thus,$z'$is a coincidence point of ST and J. Since, the view of weakly compatibility of the pair(ST, J),one has $STz = ST(Jz') = J(STz') = Jz$, which shows that$STz = Jz$.
Further,

$$d(ABx_{2n}, STz) \leq \alpha_1 \left[\frac{d(ABx_{2n}, Jz).\, d(Jz, STz) + d(STz, Ix_{2n}).\, d(Ix_{2n}, ABx_{2n})}{d(ABx_{2n}, Jz) + d(STz, Ix_{2n})}\right] +\alpha_2[d(ABx_{2n}, Ix_{2n}) + d(STz, Jz)]+\alpha_3 d(Ix_{2n}, Jz) +F\big(d(Ix_{2n}, STz).\, d(Ix_{2n}, ABx_{2n})\big)$$

which on using (1), letting $n \to \infty$, and using $STz = Jz$, reduces to

$$d(z, STz) \leq \alpha_1 \left[\frac{d(z, Jz).\, d(Jz. STz) + d(STz, z).\, d(z, z)}{d(z, Jz) + d(STz, z)}\right] + \alpha_2[d(z, z) + d(Jz, STz)] +\alpha_3 d(z, Jz) + F\big(d(STz, z).\, d(z, z)\big)$$

which implies that,

$$d(z, STz) \leq \alpha_3 d(z, STz)$$

This gives,$STz = z = Jz$. It follows from the upper part.
Again, since$ST(X) \subset I(X)$ there always exist a point $z"$in X, such that $Iz" = z$.
Thus,

$$d(ABz", z) = d(ABz", STz)$$

$$\leq \alpha_1 \left[\frac{d(ABz", Jz).\, d(Jz, STz) + d(STz, Iz").\, d(Iz", ABz")}{d(ABz", Jz) + d(STz, Iz")}\right] +\alpha_2[d(ABz", Iz") + d(Jz, STz)]+\alpha_3 d(Iz", Jz) +F\big(d(STz, Iz").\, d(Iz", ABz")\big)$$

$$= \alpha_1 \left[\frac{d(ABz", z).\, d(z, z) + d(z, z).\, d(z, ABz")}{d(ABz", z) + d(z, z)}\right] +\alpha_2[d(ABz", z) + d(z, z)] +\alpha_3 d(z, z) + F\big(d(z, z).\, d(z, ABz")\big)$$

$$= \alpha_2 d(ABz", z) + F(0)$$

equivalently, $d(ABz", z) \leq \alpha_2 d(ABz", z)$.

Which shows that$ABz" = z$.
Also, since(AB, I) are compatible and hence weakly compatible, we obtain

$$d(ABz, Iz) = d\big(AB(Iz), \mathrm{I(ABz)}\big)$$

$$\leq d(Iz, \mathrm{ABz}) = d(z, z) = 0$$

Therefore, $ABz = Iz = z$.

Thus, we have proved that z is a common fixed point of AB, ST, I and J.

Instead of AB or I, if mappings ST or J is continuous, then the proof that z is a common fixed point of AB, ST, I and J is similar.

To show that z is unique, let$v$ be the another fixed point of I, J, AB and ST.Then,

$$d(z, v) = d(ABz, STv)$$

$$\leq \alpha_1 \left[\frac{d(ABz, Jv).\, d(Jv, STv) + d(STv, Tz).\, d(Iz, ABz)}{d(ABz, Jv) + d(STv, Iz)}\right] +\alpha_2[d(ABz, Iz) + d(Jv, STv)] +\alpha_3 d(Iz, Jv) + F\big(d(STv, Iz).\, d(Iz, ABz)\big)$$

or, $d(z, v) = \alpha_3 d(z, v) + F(0)$

or, $d(z, v) \leq \alpha_3 d(z, v)$.

yielding thereby$z = v$.

Finally, we need to show that z is also a common fixed point of A,B,S,T,I and J. For this, let z be the unique common fixed point of both the pairs (AB, I) and (ST, J).Then,

$$Az = A(ABz) = A(BAz) = AB(Az)$$
$$Az = A(Iz) = I(Az)$$
$$Bz = B(ABz) = B\big(A(Bz)\big) = BA(Bz) = AB(Bz)$$
$$Bz = B(Iz) = I(Bz)$$

which show that Az and $Bz$are common fixed points of (AB, I) yielding thereby $Az=z=Bz=Iz=ABz$ in the view of uniqueness of common fixed point of the pair (AB, I).

Similarly, using the commutativity of $(S,T),(S,J)$ and (T,J) it can be shown that $Sz = z = Tz = Jz = STz$.

Now we need to show that $Az = Sz$ $(Bz = Tz)$, also remains a common fixed point of both the pairs (AB, I) and (ST, J). For this,

$$d(Az, Sz) = d(A(BAz), S(TSz))$$
$$= d(AB(Az), ST(Sz))$$
$$\leq \alpha_1\left[\frac{d\big(AB(Az), J(Sz)\big).\,d\big(J(Sz), ST(Sz)\big) + d\big(ST(Sz), I(Az)\big).\,d\big(I(Az), AB(Az)\big)}{d\big(AB(Az), J(Sz)\big) + d\big(ST(Sz), I(Az)\big)}\right]$$
$$+\alpha_2\big[d\big(AB(Az), I(Az)\big) + d\big(J(Sz), ST(Sz)\big)\big] + \alpha_3 d\big(I(Az), J(Sz)\big)$$
$$+F\left(d\big(ST(Sz), I(Az)\big).\,d\big(I(Az), AB(Az)\big)\right)$$

implies that$d(Az, Sz) = 0$ as $d\big(AB(Az), J(Sz)\big) + d\big(ST(Sz), I(Az)\big) = 0$
Using condition (3), yielding $Az = Sz$.
Similarly, it can be shown that $Bz = Tz$.
Thus, z is the unique common fixed point of A,B,S,T,I, and J

Case – II :
Suppose that$d(ABx, Jy) + d(STy, Ix) = 0$ implies that$d(ABx, STy) = 0$.then we argue as follows:
Suppose that there exists an n such that $z_n = z_{n+1}$.
Then, also $z_{n+1} = z_{n+2}$, suppose not. Then from (4) we have

$$0 < d(z_{n+1}, z_{n+2}) \leq kd(z_{n+1}, z_n)$$

yielding thereby$z_{n+1} = z_{n+2}$.

Thus, $z_n = z_{n+k}$ for$k = 1,2, \ldots$Then it follows that there exist two points w_1 and w_2 such that$v_1 = ABw_1 = Jw_2$ and $v_2 = STw_2 = Iw_1$.
Since, $d(ABw_1, Jw_2) + d(STw_2, Iw_1) = 0$,then from (2.4),

$$d(ABw_1, STw_2) = 0 \text{ i.e } v_1 = ABw_2 = STw_2 = v_2$$

Also note that, $Iv_1 = I(ABw_1) = AB(Iw_1) = ABv_2$
Similarly, $STv_2 = Jv_2$.
Define,$y_1 = ABv_1, y_2 = STv_2$,
Since$d(ABv_1, Jv_2) + d(STv_2, Jv_1) = 0$, it follows from (5.3) that$d(ABv_1, STv_2) = 0$ i.e.$y_1 = y_2$.
Thus, $ABv_1 = Iv_1 = STv_2 = Jv_2$.
But$v_1 = v_2$.

Therefore, AB,I,ST and J have a common coincidence points.
Define$w = ABv_1$, it then follows that w is also a common coincidence point of AB,I, STand J.If

$$ABw \neq ABv_1 = STv_1 \text{ then } d(ABw, STv_1) > 0.$$

But, since$d(ABw, Jv) + d(STv_1, Iw) = 0$, it follows from (5.3) that $d(ABw, STv_1) = 0$, i.e $ABw = STv_1$, a contradiction. Therefore, $ABw = STv_1 = w$ and w is a common fixed point of AB, ST, I and J.

The other part is identical to the case (I), hence it is omitted, this completes the proof.

If$F(t) = 0$, for all $t \in R^{+}$ in Theorem 3.2, we get the following result.

Theorem3.3:
Let(X, d) be a cone metric space and P a normal cone with normal constant K. Suppose mappings $A, B, S, T, I, J: X \to X$, satisfying$AB(X) \subset J(X)$, $ST(X) \subset I(X)$ and for each$x, y \in X$ either,

$$d(ABx, STy) \leq \alpha_1 \left[\frac{d(ABx, Jy). d(Jy, STy) + d(STy, Ix). d(Ix, ABx)}{d(ABx, Jy) + d(STy, Ix)}\right] + \alpha_2[d(ABx, Ix) + d(Jy, STy)] + \alpha_3 d(Ix, Jy) \quad \ldots (5)$$

if $d(ABx, Jy) + d(STy, Ix) \neq 0, \alpha_i \geq 0$ $(i = 1,2,3 \ldots)$ with atleast one α_i non zero and$\alpha_1 + 2\alpha_2 + \alpha_3 \leq 1$

$$d(ABx, STy) = 0 \text{ if } d(ABx, Jy) + d(STy, Ix) = 0 \quad \ldots (6)$$

if either
(a)(*AB*, *I*) are compatible, *I* or *AB* is continuous and (*ST*,*J*) are weakly compatible or(a') (*ST*,*J*) are compatible, *J* or *ST* is continuousthen,*AB*,*ST*, *I* and *J* have a unique common fixed point. Furthermore, if the pairs $(A, B), (A, I), (B, I), (S, T), (S, J)$ and (T, J) are commuting mappings then *A*,*B*,*S*,*T* ,*I* and *J* have a unique common fixed point.

Putting$\alpha_2 = 0$, $AB = A$ and $ST = B$,this will give the following generalization of Jeong-Rhoades [8] in cone metric spaces.

Corollary 3.4: Let(X, d) be a cone metric space and P a normal cone with normal constant K. Suppose mappings $A, B, S, T: X \to X$satisfying$A(X) \subset T(X)$, $B(X) \subset S(X)$ and for each$x, y \in X$ either

$$d(Ax, By) \leq \alpha \left[\frac{d(Ax, Sx). d(Sx, By) + d(By, Ty). d(Ty, Ax)}{d(Sx, By) + d(Ty, Ax)}\right] + \beta d(Sx, Ty)$$

If$d(Sx, By) + d(Ty, Ax) \neq 0$ where $\alpha, \beta \geq 0, \alpha + \beta < 1$
or, $d(Ax, By) = 0$ if $d(Sx, By) + d(Ty, Ax) = 0$

if either
(a) (*A*, *S*) are compatible, *A* or *S* is continuous and (*B*,*T*) are weakly compatible or(a')(*B*,*T*) are compatible, *B* or *T* is continuous and (*A*,*S*) are weakly compatible, then *A*,*B*,*S* and *T* have a unique common fixed point *z*. Moreover,*z* is the unique common fixed point of *A* and *S* and of *B* and *T*.

REFERENCES

1. Abbas, M. and Jungck, G., "Common fixed point results for noncommitting mappings without continuity in cone metric spaces", J. Math. Anal. Appl., 341(1), 416–420, 2008.
2. Abbas, M. and Rhoades, B.E., "Fixed and periodic results in cone metric spaces", Applied Mathematics Letters, 22(4), 511–515, 2009.
3. A. K. Goyal, "Common fixed point theorem for six mappings in complete metric spaces", Bull. Pure Appl. Math., 3 (1), 24–35, 2009.
4. A. K. Goyal., "Common fixed point theorems for weakly compatible mappings satisfying rational contractive conditions", International Journal of Psychosocial Rehabilitation, Vol. 17(1),138-145, 2013
5. Han, Y. and Xu, S., "Some fixed point theorems for expanding mappings without continuity in cone metric spaces", Fixed Point Theory Appl., Vol. 03, 1–9, 2013.
6. Huang, G. and Zhang, X., "Cone metric spaces and fixed point theorems of contractive mappings", J. Math. Anal. Appl., 332,1468–1476, 2007.
7. Ilic, D. and Rakocevic, V., "Common fixed points for maps on cone metric spaces", J. Math. Anal. Appl., 341(2),876–882, 2008.
8. G.S. Jeong and B.E. Rhoades., "Some remarks for improving fixed point theorem for more than two maps", Ind. J. Pure. Appl. Math ***28 (9) (1997)***, 1177-1196.
9. G. Jungck., "Commuting mappings and fixed points", Amer. Math. Monthly., ***83 (1976)***, 261-263.
10. G. Jungck., "Compatible mappings and common fixed point", Internet. J. Math and Math. Sci., ***9 (4) (1986)***, 771-779.

11. G. Jungck and B.E. Rhoades., "Fixed point for set valued function without continuity", Ind. J. Pure. Appl. Math., ***29 (3) (1998)***, 227-238.
12. Slobodan, C. Nesic., "Results on fixed points of aymptotically regular mappings", Ind. J. Pure. Appl. Math., ***30 (5) (1999),*** 491-494.
13. S. Sessa., "On a weak commutativity condition in fixed point consideration", Publ. Inset. Math., ***32 (1982)***, 149-153.