Common Fixed Point Theorems For Weakly Compatible Mappings Satisfying Rational Inequality In Cone Metric Spaces

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Abstract
Jungck and Rhoades [13] introduced the notion of weakly compatible mappings, which is weaker than compatibility. Many interesting fixed point theorems for weakly compatible maps satisfying contractive type conditions have been obtained by various authors. In this paper, we prove a common fixed point theorem for three pairs of weakly compatible mappings satisfying a rational inequality without any continuity requirement which generalize several previously known results due to Imdad and Ali [3], Goyal([3], [4]),Goyal and Gupta ([5], [6]),Imdad-Khan [4], Jeong-Rhoades [10] and others.

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1. INTRODUCTION AND PRELIMINARIES:

Huang and Zhang [8] generalized the concept of a cone metric space, re-placing the set of real numbers by an ordered Banach space and obtained some common fixed point theorems for mappings satisfying different contractive conditions over cone metric space. Subsequently, Abbas and Jungck[1] and Abbas and Rhoades[2] studied common fixed point theorems in cone metric spaces. Moreover, Huang and Zhang [8], Abbas and Jungck[1], IIlic and Rakocevic [9] proved their results for normal cones. Jungck [12] generalized the concept of weak commuting by defining the term compatible mappings and proved that the weakly commuting mappings are compatible but the converse is not true. Jungck and Rhoades [13] defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space utilizing these concepts. In this paper, we prove a common fixed point theorem for three pairs of weakly compatible mappings satisfying a rational inequality without any continuity requirement in complete cone metric spaces. Our work generalizes some earlier results of Imdad and Ali [3], Nesic [18], Jeong and Rhoades [10], Goyal ([3], [4]),Goyal and Gupta ([5], [6]). Someexamples are also furnished to demonstrate the validity of the hypothesis.

Definition 1.1: Let E be a real Banach space and P be a subset of E. The subset P is called a cone if and only if

(a) P is closed, non-empty and $P \neq \{0\}$
(b) $a, b \in R,\ a, b \geq 0,\ x, y \in P$ implies $ax + by \in P$
(c) $x \in P$ and $-x \in P \Rightarrow x = 0$ i.e $P \cap (-P) = \{0\}$

Definition 1.2: Let P be a cone in a Banach space E i.e. given a cone $P \subset E$, define partial ordering '$\leq$' with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$while $x \ll y$ will stand for $y - x \in Int\ P$, where $Int\ P$ denote the interior of the set P. This cone P is called an order cone.

The following definitions are in literature of Huang and Zhang [8].

Definition 1.3: Let E be a real Banach space and $P \subset E$ be an order cone. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,
$0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.4: Let X be a non-empty and E be a real Banach space. Suppose that the mapping $d: X \times X \to E$ satisfies

(d1) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$

(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$

(d3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$

Then d is called a cone metric on X and (X,d) is called a cone metric space.

It is obvious that cone metric spaces generalize metric spaces because each metric space is a cone metric space with $E = R$ and $P = [0, +\infty[$

Example 1.5: (a) Let $E = R^2$, $P = \{(x,y) \in E\, |x,y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \to E$ such that

$d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant.

Then (X,d) is a cone metric space.

(b) Let $E = R^n$ with $P = \{(x_i, \dots, x_n) : x_i \geq 0, \forall i = 1,2, \dots, n\} X = R$ and $d: X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha_i|x-y|, \dots, \alpha_{n-1}|x-y|)$

where $\alpha_i \geq 0$ for all $1 \leq i \leq n-1$. Then (X,d) is a cone metric space.

Definition 1.6: Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$is a

(a) convergent sequence or $\{x_n\}$ converges to x if for every c in E with $c \gg 0$, there is an $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) \ll c$ for some fixed point x in X where x is that limit of $\{x_n\}$. This is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$. Completeness is defined in the standard way.

It was proved in [7] if (X,d) be a cone metric space, P be a normal cone with normal constant K and $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(b) Cauchy sequence if for c in E with $c \gg 0$, there is an $n_0 \in N$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$.

It was proved in [7] if (X,d) be a cone metric space, P be a normal cone with normal constant K and $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequnence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 1.7: A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence in unique provided P is a normal cone with normal constant K.

In recent years several definitions of conditions weaker than commutativity have appeared which facilitated significantly to extend the Jungck's[12] theorem and several others. Foremost among of them is perhaps the weak commutativity condition introduced by Sessa [18] which can be described as follows:

Definition 1.8: Let S and T be mappings from a cone metric space (X,d) into itself. Then S and T are said to be weakly commuting mappings on X if

$d(STx, TSx) \leq d(Sx, Tx)$, for all $x \in X$.

obviously a commuting pair is weakly commuting but its converse need not be true as is evident from the following example.

Example 1.9: Consider the set $X = [0,1]$ with the usual metric defined by

$$d(x,y) = |x-y| = \|x-y\|$$

Define S and $T: X \to X$ by

$Sx = \frac{x}{3-2x}$ and $Tx = \frac{x}{3}$ for all $x \in X$.

Then, we have to any x in X

$STx = \frac{x}{9-2x}$ and $TSx = \frac{x}{9-6x}$

Hence $ST \neq TS$. Thus, S and T do not commute.

$$\text{Again, } d(STx, TSx) = \left\|\frac{x}{9-2x} - \frac{x}{9-6x}\right\| = \frac{4x^2}{(9-2x)(9-6x)} \leq \frac{2x^2}{3(3-2x)} = \frac{x}{3-2x} - \frac{x}{3} = d(Sx, Tx)$$

and thus S and T commute weakly.

Example 1.10: Consider the set $X = [0,1]$ with the usual metric $d(x,y) = \|x-y\|$. Let $Sx = \frac{x}{2}$ and $Tx = \frac{x}{2+x}$, for every $x \in X$. Then, for all $x \in X$

$STx = \frac{x}{4+2x}$ and $TSx = \frac{x}{4+x}$

Hence, $ST \neq TS$. Thus, S and T do not commute.

$$\text{Again, } d(STx, TSx) = \left\|\frac{x}{4+2x} - \frac{x}{4+x}\right\| = \frac{x^2}{(4+x)(4+2x)}$$

$$\leq \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x} = d(Sx, Tx)$$

and thus, S and T commute weakly.
Obviously, the class of weakly commuting is wider and includes commuting mappings as subclass.
Jungck [12] has observed that for $X = R$ if $Sx = x^3$ and $Tx = 2x^3$ then S and T are not weakly commuting. Thus it is desirable to a less restrictive concept which he termed as 'compatibility' the class of compatible mappings is still wider and includes weakly commuting mappings as subclass as is evident from the following definition of Jungck [12].

Definition 1.11: Let S and T be self mappings on a cone metric space (X, d). Then S and T are said to be compatible mappings on X if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some point $t \in X$.

Obviously, any weakly commuting pair $\{S, T\}$ is compatible, but the converse is not necessarily true, as in the following example.
Example 1.12: Let $Sx = x^3$ and $Tx = 2x^3$ with $X = R$ with the usual metric. Then S and T are compatible, since $|Tx - Sx| = |x^3| \to 0$ if and only if $|STx - TSx| = 6|x^9| \to 0$ But $|STx - TSx| \leq |Tx - Sx|$ is not true for all $x \in X$, say for example at $x = 1$.

Definition 1.13: Let S and T be self maps of a set X. If $w = Sx = Tx$ for some x in X, then x is called a coincidence point of S and T and w is called a point of coincidence of S and T.

Definition 1.14: A pair of self mappings (S, T) on a cone metric space (X, d) is said to be weakly compatible if the mappings commute at their coincidence points i.e $Sx = Tx$ for some $x \in X$ implies that $STx = TSx$.

Example 1.15: Let X = [2, 20] with usual metric define

$$Tx = \begin{cases} 2 & if \quad x = 2 \\ 12 + x & if \quad 2 < x \leq 5 \\ x - 3 & if \quad 5 < x \leq 20 \end{cases} \quad and \quad Sx = \begin{cases} 2 & if \quad x \in \{2\} \cup (5,20] \\ 8 & if \quad 2 < x \leq 5 \end{cases}$$

S and T are weakly compatible mappings which is not compatible.

Remark 1.16: Let (X, d) be a cone metric space with a cone P. If $d(x, y) \leq hd(x, y)$ for all $x, y \in X, h \in (0,1)$, then $d(x, y) = 0$, which implies that $x = y$.

2. MAIN RESULTS

Let R^+ be the set of non-negative real numbers, and let $F: R^+ \to R^+$ be a mapping such that $F(0) = 0$ and F is continuous at 0.

Motivated by the contractive condition given by, Jeong Rhoades [10] and Nesic [16]we prove the following theorem.

Theorem 2.1: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Let A, B, S, T, I and J be self-mappings of a cone metric space (X, d) satisfying $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ such that for each $x, y \in X$ either

$$d(ABx, STy) \leq \beta_1 \left[\frac{\{d(ABx, Ix)\}^2 + \{d(STy, Jy)\}^2}{d(ABx, Ix) + d(STy, Jy)}\right] + \beta_2 d(Ix, Jy) + \beta_3[d(ABx, Jy) + d(STy, Ix)] + F(min\{d^2(Ix, Jy), d(Ix, ABx).d(Ix, STy), d(Jy, STy).d(Jy, ABx)\}) \quad \dots (1)$$

if $d(ABx, Ix) + d(STy, Jy) \neq 0$, $\beta_i \geq 0$ $(i = 1,2,3)$ with at least one β_i non zero and $2\beta_1 + \beta_2 + 2\beta_3 < 1$
or, $d(ABx, STy) = 0$ if $d(ABx, Ix) + d(STy, Jy) = 0$ … (2)

If one of the $AB(X)$, $ST(X)$, $J(X)$ and $I(X)$ is a complete subspace of X, then
(a) (AB, I) has a coincidence point
(b) (ST, J) has a coincidence point

Further, if the pairs (AB, I) and (ST, J) are coincidentally commuting (weakly compatible), then AB, ST, I and J have a unique common fixed point. Moreover, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J) and (T,J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point. Since $AB(X) \subset J(X)$, we can choose a point x_1 in X such that $ABx_0 = Jx_1$. Again, since $ST(X) \subset I(X)$, we can choose a point x_2 in $X$with $STx_1 = Ix_2$. Using this process repeatedly, we can construct a sequence $\{z_n\}$ such that
$z_{2n} = ABx_{2n} = Jx_{2n+1}$ and $z_{2n+1} = STx_{2n+1} = Ix_{2n+2}$for $n = 0, 1, 2, ...$

Now, we consider two cases

Case I: If $d(ABx, Ix) + d(STy, Jy) \neq 0$. Then on using inequality (1), we have

$$d(z_{2n+1}, z_{2n+2}) = d(STx_{2n+1}, ABx_{2n+2})$$

$$\begin{aligned}
&\leq \beta_1 \left[\frac{\{d(ABx_{2n+2}, Ix_{2n+2})\}^2 + \{d(STx_{2n+1}, Jx_{2n+1})\}^2}{d(ABx_{2n+2}, Ix_{2n+2}) + (STx_{2n+1}, Jx_{2n+1})}\right] + \beta_2 d(Ix_{2n+2}, Jx_{2n+1}) \\
&\quad + \beta_3 [(d(ABx_{2n+2}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n+2})] \\
&\quad + F[min\{d^2(Ix_{2n+2}, Jx_{2n+1}), \\
&\qquad d(Ix_{2n+2}, ABx_{2n+2}).d(Ix_{2n+2}, STx_{2n+1}), \\
&\qquad d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, ABx_{2n+2})\}] \\
&\leq \beta_1 \frac{[d(ABx_{2n+2}, Ix_{2n+2}) + d(STx_{2n+1}, Jx_{2n+1})]^2}{d(ABx_{2n+2}, Ix_{2n+2}) + d(STx_{2n+1}, Jx_{2n+1})} + \beta_2 d(Ix_{2n+2}, Jx_{2n+1}) \\
&\quad + \beta_3 [d(ABx_{2n+2}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n+2})] \\
&\quad + F[min\{d^2(Ix_{2n+2}, Jx_{2n+1}), \\
&\qquad d(Ix_{2n+2}, ABx_{2n+2}).d(Ix_{2n+2}, STx_{2n+1}), \\
&\qquad d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, ABx_{2n+2})\}] \\
&\leq \beta_1 [d(ABx_{2n+2}, Ix_{2n+2}) + \mathrm{d}(STx_{2n+1}, Jx_{2n+1})] \\
&\quad + \beta_2 d(Ix_{2n+2}, Jx_{2n+1}) \\
&\quad + \beta_3 [d(ABx_{2n+2}, Jx_{2n+1}) + \mathrm{d}(STx_{2n+1}, Ix_{2n+2})] \\
&\quad + F[min\{d^2(Ix_{2n+2}, Jx_{2n+1}), \\
&\qquad d(Ix_{2n+2}, ABx_{2n+2}).d(Ix_{2n+2}, STx_{2n+1}), \\
&\qquad d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, ABx_{2n+2})\}] \\
&\leq \beta_1 [d(z_{2n+2}, z_{2n+1}) + \mathrm{d}(z_{2n+1}, z_{2n})] + \beta_2 d(z_{2n+1}, z_{2n}) \\
&\quad + \beta_3 [d(z_{2n+2}, z_{2n+1}) + d(z_{2n+1}, z_{2n})] \\
&\quad + F[min\{d^2(z_{2n+1}, z_{2n}), d(z_{2n+1}, z_{2n+2}).d(z_{2n+1}, z_{2n+1}), \\
&\qquad d(z_{2n}, z_{2n+1}).d(z_{2n}, z_{2n+2})\}] \\
&\leq (\beta_1 + \beta_3)\, d(z_{2n+2}, z_{2n+1}) + (\beta_1 + \beta_2 + \beta_3) d(z_{2n}, z_{2n+1}) \\
&\quad + F[min\{d^2(z_{2n+1}, z_{2n}), \\
&\qquad d(z_{2n+1}, z_{2n+2}).0, d(z_{2n}, z_{2n+1}).\ d(z_{2n}, z_{2n+2})\}]
\end{aligned}$$

or, $$\begin{aligned}
d(z_{2n+2}, z_{2n+1}) &\leq \left(\frac{\beta_1 + \beta_2 + \beta_3}{1 - \beta_1 - \beta_3}\right) d(z_{2n+1}, z_{2n}) \\
&\quad + \frac{1}{(1 - \beta_1 - \beta_3)} F[min\{d^2(z_{2n+1}, z_{2n}), \\
&\qquad 0, d(z_{2n}, z_{2n+1}).d(z_{2n}, z_{2n+2})\}] \\
&= \left(\frac{\beta_1 + \beta_2 + \beta_3}{1 - \beta_1 - \beta_3}\right) d(z_{2n+1}, z_{2n}) + \frac{1}{(1 - \beta_1 - \beta_3)} F(0)
\end{aligned}$$

or, $$d(z_{2n+1}, z_{2n+2}) \leq \left(\frac{\beta_1 + \beta_2 + \beta_3}{1 - \beta_1 - \beta_3}\right) d(z_{2n}, z_{2n+1}) + 0 \quad [\because F(0) = 0]$$

or, $$d(z_{2n+1}, z_{2n+2}) \leq \left(\frac{\beta_1 + \beta_2 + \beta_3}{1 - \beta_1 - \beta_3}\right) d(z_{2n}, z_{2n+1})$$

Following the same process, we can show that

$$d(z_{2n}, z_{2n+1}) \leq \left(\frac{\beta_1 + \beta_2 + \beta_3}{1 - \beta_1 - \beta_3}\right) d(z_{2n-1}, z_{2n})$$

Thus, for every n, we can show that

$$d(z_n, z_{n+1}) \le \alpha d(z_{n-1}, z_n) \qquad \dots (3)$$

Where $\alpha = \frac{\beta_1+\beta_2+\beta_3}{1-\beta_1-\beta_3} < 1$

Now, by induction

$$\begin{aligned} d(z_n , z_{n+1}) &\le \alpha d(z_{n-1} , z_n) \\ &\le \alpha^2 d(z_{n-2} , z_{n-1}) \\ &\vdots \\ &\le \alpha^n d(z_0 , z_1) \end{aligned}$$

For any $m > n$, we get,

$$\begin{aligned} d(z_n, z_m) &\le d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + \dots + d(z_{m-1}, z_m) \\ &\le [\alpha^n + \alpha^{n+1} + \dots\dots\dots + \alpha^{m-1}] d(z_0 , z_1) \\ &\le \frac{\alpha^n}{1-\alpha} d(z_0 , z_1) \end{aligned}$$

Now, using normality of cone, we get

$$||d(z_n , z_m)|| \le \frac{\propto^n}{1-\propto} . K ||d(z_0 , z_1)||$$

This implies that $d(z_n, z_m) \to 0$ as $n, m \to \infty$

Hence, sequence $\{z_n\}$ described by

$$\{ABx_0, STx_1, ABx_2, \dots STx_{2n-1}, ABx_{2n}, STx_{2n+1}, \dots\}$$

is a Cauchy sequence in a cone metric space (X, d). Now, let $ST(X)$ is a complete subspace of X, then the subsequence $\{z_{2n+1}\}$ which is contained in $ST(X)$ also get a limit z in $ST(X)$ i.e.

$$\lim_{n\to\infty} ST\, x_{2n+1} = z$$

Since, $ST(X) \subset I(X)$, there exists a point $z' \in X$ such that $Iz' = z$.

Again, as $\{z_n\}$ is a Cauchy sequence containing a convergent subsequence $\{z_{2n+1}\}$, therefore the sequence $\{z_n\}$ also converges which implies the convergence of $\{z_{2n}\}$ being a subsequence of the convergent sequence $\{z_n\}$ i.e. $\lim_{n\to\infty} J\, x_{2n+1} = z$.

To prove that $ABz' = z$ put $x = z'$ and $y = x_{2n-1}$ in (1), we get

$$\begin{aligned} d(ABz', STx_{2n-1}) \le{}& \beta_1 \left[\frac{\{d(ABz', Iz')\}^2 + \{d(STx_{2n-1}, Jx_{2n-1})\}^2}{d(ABz', Iz) + d(STx_{2n-1}, Jx_{2n-1})}\right] + \beta_2 d(Iz', Jx_{2n-1}) \\ &+ \beta_3 [d(ABz', Jx_{2n-1}) + d(STx_{2n-1}, Iz')] \\ &+ F[min\{d^2(Iz', Jx_{2n-1}), \\ &\quad d(Iz', ABz').d(Iz', STx_{2n-1}), \\ &\quad d(Jx_{2n-1}, STx_{2n-1}).d(Jx_{2n-1}, ABz')\}] \end{aligned}$$

on letting $n \to \infty$, above reduces to

$$\begin{aligned} d(ABz', z) \le{}& \beta_1 \left[\frac{\{d(ABz', z)\}^2 + \{d(z, z)\}^2}{d(ABz', z) + d(z, z)}\right] + \beta_2 d(z, z) + \beta_3 [d(ABz', z) + d(z, z)] \\ &+ F[min\{d^2(z, z), d(z, ABz').d(z, z), d(z, z).d(z, ABz')\}] \\ \le{}& \beta_1 d(ABz', z) + \beta_3 d(ABz', z) \\ &+ F[min\{0, d(z, ABz').0, 0.d(z, ABz')\}] \\ \le{}& (\beta_1 + \beta_3) d(ABz', z) + F(0) \end{aligned}$$

or, $\quad d(ABz', z) \le (\beta_1 + \beta_3)\, d(ABz', z) \qquad [\because F(0) = 0]$

which gives $ABz' = z$ [by using Remark (1.16)].

Thus, we get $ABz' = Iz' = z$ and result (a) is established i.e the pair (AB, I) has a coincidence point.

Since z is in the range of AB i.e. $ABz' = z$ and $AB(X) \subset J(X)$ there always exists a point z" such that $Jz" = z$

Now, $d(z, STz") = d(ABz', STz")$

$$\leq \beta_1 \left[\frac{\{d(ABz', Iz')\}^2 + \{d(STz,\ Jz)\}^2}{d(ABz', Iz') + d(STz", Jz")}\right] + \beta_2 d(Iz', Jz")$$
$$+\beta_3 d(ABz', Jz") + d(STz", Iz')$$
$$+F[min\{d^2(Iz', Jz"), d(Iz', ABz').d(Iz', STz"),$$
$$d(Jz", STz").d(Jz", ABz')\}]$$
$$\leq \beta_1 \left[\frac{\{d(z,z)\}^2 + \{d(STz", z)\}^2}{d(z,z) + d(STz", z)}\right] + \beta_2 d(z,z) + \beta_3 [d(z,z) + d(STz", z)]$$
$$+F[min\{d^2(z,z), d(z,z).d(z, STz"), d(z, STz").d(z,z)\}]$$
$$\leq (\beta_1 + \beta_3)\ d(z, STz") + F[min\{0,0,0\}]$$

or, $d(z, STz") \leq (\beta_1 + \beta_3) d(z, STz") + F(0)$

or, $d(z, STz") \leq (\beta_1 + \beta_3) + d(z, STz")[\because F(0) = 0]$

which implies that $STz" = z = Jz"$ i.e. the pair (ST, J) has a coincidence point.

This establishes the result (b).

If we assume that $I(X)$ is a complete subspace of X, then similar arguments establish results (a) and (b). The remaining two cases pertain essentially to the previous cases.

Infact, if $ST(X)$ is complete then $z \in ST(X) \subset I(X)$ and if $AB(X)$ is complete, then,

$$z \in AB(X) \subset J(X).$$

Thus, the results (a) and (b) are completely established.

Furthermore, if the pairs (AB, I) and (ST, J) are coincidentally commuting at z' and $z"$ respectively then

(i) $z = ABz' = Iz' = STz" = Jz"$

(ii) $ABz = AB(Iz') = I(ABz') = Iz$

(iii) $STz = ST(Jz") = J(STz") = Jz$

Since, $d(ABz', Iz') + d(STz, Jz) = 0$

Therefore, by (2), we get $d(ABz', STz) = d(z, STz) = 0$

or, $z = STz$.

Similarly, $d(ABz, Iz) + d(STz", Jz") = 0$, therefore by (2), we get

$$d(ABz, STz") = d(ABz, z) = 0$$

or, $z = ABz$.

Thus, $ABz = Iz = STz = Jz = z$, which shows that z is a common fixed point of AB, ST, I and J.

To show that z is unique, let u be another fixed point of I, J, AB and ST. Then,

$$d(z,u) = d(ABz, STu)$$
$$\leq \beta_1 \left[\frac{\{d(ABz, Iz)\}^2 + \{d(STu, Ju)\}^2}{d(ABz, Iz) + d(STu, Ju)}\right] + \beta_2 d(Iz, Ju)$$
$$+\beta_3 [d(ABz, Ju) + d(STu, Iz)]$$
$$+F[min\{d^2(Iz, Ju), d(Iz, ABz).d(Iz, STu),$$
$$d(Ju, STu).d(Ju, ABz)\}]$$
$$\leq \beta_1 \left[\frac{\{d(ABz, Iz) + d(STu, Ju)\}^2}{d(ABz, Iz) + d(STu, Ju)}\right] + \beta_2 d(Iz, Ju)$$
$$+\beta_3 [d(ABz, Ju) + d(STu, Iz)] \quad +F[min\{d^2(Iz, Ju), d(Iz, ABz).d(Iz, STu),$$
$$d(Ju, STu).d(Ju, ABz)\}]$$
$$\leq \beta_1 [d(ABz, Iz) + d\ (STu, Ju)] + \beta_2 d(Iz, Ju)$$
$$+\beta_3 [d(ABz, Ju) + d(STu, Iz)]$$
$$+F[min\{d^2(Iz, Ju), d(Iz, ABz).d(Iz, STu),$$
$$d(Ju, STu).d(Ju, ABz)\}]$$
$$\leq (\beta_2 + 2\beta_3) d(z,u)$$
$$+F[min\{d^2(z,u), d(z,z).d(z,u), d(u,u).d(u,z)\}]$$
$$\leq (\beta_2 + 2\beta_3) d(z,u) + F[min\{d^2(z,u), 0, 0\}]$$
$$\leq (\beta_2 + 2\beta_3) d(z,u) + F(0)$$

$$\leq (\beta_2+2\beta_3)d(z,u) \qquad [\because F(0)=0]$$

yielding, thereby $z = u$.

Thus, z is a unique common fixed point of AB, ST, I and J.
Finally, we prove that z is also a common fixed point A,B, S,T, I and J. For this, let both the pairs (AB, I) and (ST, J) have a unique common fixed point z.

Then $Az = A(ABz) = A(BAz) = AB(Az)$

$Az = A(Iz) = \mathrm{I(Az)}$

$Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz)$

$Bz = B(Iz) = I(Bz)$

which shows that (AB, I) has common fixed points, which are Az and Bz. We get thereby, $Az = z = Bz = Iz = ABz$, by virtue of uniqueness of common fixed point of pair (AB, I).
Similarly, using the commutativity of (S,T), (S,J) and (T,J),

$Sz = z = Tz = Jz = STz$ can be shown.

Now, to show that Az = Sz (Bz = Tz), we have

$$d(Az,Sz) = d\big(A(BAz),S(TSz)\big) = d(AB(Az),ST(Sz))$$
$$\leq \beta_1\left[\frac{\{d(AB(Az),I(Az))\}^2+\{d(ST(Sz),J(Sz))\}^2}{d(AB(Az),I(Az))+d(ST(Sz),J(Sz))}\right]$$
$$+\beta_2 d\big(I(Az),J(Sz)\big)$$
$$+\beta_3\big[d\big(AB(Az),J(Sz)\big)+d\big(I(Az),ST(Sz)\big)\big]$$
$$+F\big[min\{d^2\big(I(Az),J(Sz)\big),$$
$$d\big(I(Az),AB(Az)\big).d\big(I(Az),ST(Sz)\big),$$
$$d\big(J(Sz),ST(Sz)\big).d\big(J(Sz),AB(Az)\big)\}\big]$$

which implies that $d(Az,Sz) = 0$
(as $d\big(AB(Az),I(Az)\big) + d\big(ST(Sz),J(Sz)\big) = 0$), using condition (2), thereby we get $Az = Sz$.
Similarly, $Bz = Tz$can be shown.
Hence, z is a unique common fixed point of A, B, S, T, I and J.

Case II: Let $d(ABx,Ix) + d(STy,Jy) = 0$ implies that $d(ABx,STy) = 0$. Then we argue as follows:
Here we show that if $y_n = y_{n+1}$ for some n, then AB, ST, I and J have a common fixed point.
Suppose that there exists as n such that $z_n = z_{n+1}$. Then also $z_{n+1} = z_{n+2}$.
For if, $z_{n+1} \neq z_{n+2}$, then from (3), with n replaced by $n+1$, we get,

$0 < d(z_{n+1},z_{n+2}) = 0$ a contradiction, gives $z_{n+1} = z_{n+2}$.

Thus, $z_n = z_{n+\alpha}$ for $\alpha = 1,2,...$

If follows that there exists two points u_1 and u_2 such that $v_1 = ABu_1 = Iu_1$ and $v_2 = STu_2 = Ju_2$. Since $d(ABu_1,Iu_1) + d(STu_2,Ju_2) = 0$ then from (2), we get
$d(ABu_1,STu_2) = 0$ i.e $v_1 = ABu_1 = STu_2 = v_2$
Also, note that $Iv_1 = I(ABu_1) = AB(Iu_1) = ABv_1$.
Similarly, $Jv_2 = J(STu_2) = ST(Ju_2) = STv_2$.
Define$y_1 = ABv_1$, $y_2 = STv_2$
Since $d(ABv_1,Iv_1) + d(STv_2,Jv_2) = 0$ it follows from (2) that

$d(ABv_1,STv_2) = 0$

or, $ABv_1 = STv_2$ i.e $y_1 = y_2$.
Thus $ABv_1 = Iv_1 = STv_2 = Jv_2$

But, $v_1 = v_2$, therefore AB, I, ST and J have a common coincidence point.

Define $u = ABv_1$, which asserts that u is also a common point of coincidence of AB, ST, I and J. If $ABu \neq ABv_1 = STv_1$, then $d(ABu, STv_1) > 0$ but since $d(ABu, Iu) + d(STv_1, Jv_1) = 0$, it follows from (2) that $(ABu, STv_1) = 0$, i.e $ABu = STv_1$ which is a contradiction. Therefore, $ABu = ABv_1 = u$ and u is a

common fixed point of AB, ST, I and J.
The rest of the proof is identical to the case(I), hence it is omitted.
This completes the proof.

If we put $F(t) = 0$ for all $t \in R^+$ in theorem (2.1), we obtain the following, which generalize the result of Imdad and Ali [3] in cone metric space for six mappings.

Corollary 2.2. Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Let A, B, S, T, I and J be self-mappings of a cone metric space (X, d) satisfying $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ such that for each $x, y \in X$ either

$$d(ABx, STy) \leq \beta_1 \left[\frac{\{d(ABx,Ix)\}^2+\{d(STy,Jy)\}^2}{d(ABx,Ix)+d(STy,Jy)}\right] + \beta_2 d(Ix, Jy)$$
$$+\beta_3[d(ABx, Jy) + d(STy, Ix)]$$

if $d(ABx, Ix) + d(STy, Jy) \neq 0$, $\beta_i \geq 0$ $(i = 1,2,3)$ with at least one β_i non zero and $2\beta_1 + \beta_2 + 2\beta_3 < 1$
or, $d(ABx, STy) = 0$ if $d(ABx, Ix) + d(STy, Jy) = 0$

If one of the $AB(X)$, $ST(X)$, $J(X)$ and $I(X)$ is a complete subspace of X, then
(a) (AB, I) has a coincidence point
(b) (ST, J) has a coincidence point

Further, if the pairs (AB, I) and (ST, J) are coincidentally commuting (weakly compatible), then AB, ST, I and J have a unique common fixed point. Moreover, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J) and (T,J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.

Putting $AB = A, ST = B$ in corollary (2.2), we obtain the following generalization of the result of Imdad and Ali [3] in cone metric space.

Corollary 2.3: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Let A, B, S and T be self-mappings of a cone metric space (X, d) with $A(X) \subset T(X)$and $B(X) \subset S(X)$ such that for each $x, y \in X$ either

$$d(Ax, By) \leq \beta_1 \left[\frac{\{d(Ax,Sx)\}^2+\{d(By,Ty\}^2}{d(Ax,Sx)+d(By,Ty)}\right] + \beta_2\, d(Sx, Ty)$$
$$+\beta_3[d(Ax, Ty) + d(Bx, Sx)]$$

If $d(Ax, Sx) + d(By, Ty) \neq 0, \beta_i \geq 0$ $(i = 1,2,3)$ (with at least one β_i non zero) and $2\beta_1 + \beta_2 + 2\beta_3 < 1$or $d(Ax, By) = 0$ whereever
$d(Ax, Sx) + d(By, Ty) = 0.$

If one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$is a complete subspace of X, then
(a) (A,S)has a coincidence point
(b) (B,T)has a coincidence point

Further, if the pairs (A,S) and (B,T) are coincidentally commuting then A, B, S and T has a unique fixed point z.
On the basis of the above corollary (2.2), we have the following result of Singh et al. [19], whose proof is similar to that of corollary (2.2).

Corollary 2.4: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Let A, B, S, T, I and J be self-mappings of a cone metric space (X, d) satisfying $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ such that for each $x, y \in X$.

$$d(ABx, STy) \leq \beta_1[d(ABx, Ix) + d(STy, Jy)] + \beta_2 d(Ix, Jy)$$

$$+\beta_3[d(ABx,Jy)+d(STy,Ix)]$$

where $\beta_i \geq 0, (i = 1,2,3)$ (with at least one β_i non zero) and $2\beta_1+\beta_2+2\beta_3 < 1$

If one of the $AB(X)$, $ST(X)$, $J(X)$ and $I(X)$ is a complete subspace of X, then
(a) (AB, I) has a coincidence point
(b) (ST, J) has a coincidence point

Further, if the pairs (AB, I) and (ST, J) are coincidentally commuting (weakly compatible), then AB, ST, I and J have a unique common fixed point.
Moreover, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J) and (T,J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.
Proof: Since

$$\frac{[d(ABx,Ix)]^2+[d(STy,Jy)]^2}{d(Ax,Fx)+d(Sy,\ Gy)} \leq \frac{[d(ABx,Ix)+d(STy,Jy)]^2}{d(Ax,Fx)+d(Sy,\ Gy)} = d(ABx,Ix)+d(STy,Jy)$$

Using above inequality in main Theorem (2.1), we get the corollary (2.4).
Taking $AB = A, ST = B, I = J = S$ in corollary (2.4), we obtain the following result of Olaleru [17].

Corollary 2.5: Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. Let A, B and S be self-mappings of a cone metric space (X,d) satisfying $A(X) \subset S(X)$, $B(X) \subset S(X)$ such that for each $x,y \in X$.

$$d(Ax,By) \leq \beta_1[d(Ax,Sx)+d(By,Sy)]+\beta_2 d(Sx,Sy)$$
$$+\beta_3[d(Ax,Sy)+d(By,Sx)]$$

where $\beta_i \geq 0, (i = 1,2,3)$ (with at least one β_i non zero) and $2\beta_1+\beta_2+2\beta_3 < 1$

If one of the $A(X), B(X)$and$S(X)$is a complete subspace of X, then the pair (AB,S)have unique coincidence point.

Further, if the pairs (A, S) and (B, S) are coincidentally commuting (weakly compatible), then A, B and S have a unique common fixed point.
Now, we furnish an example to demonstrate the validity of the hypothesis of our Corollary(2.2).

Example 2.6: Consider $X = [0,1]$ with the usual metric defined by

$d(x,y) = \|x-y\| = |x-y|$ and $F = R =$ Real Banach space.

Define self mappings A, B, S, T, I and J on X by

$$Ax = \frac{3x}{8}, Bx = \frac{4x}{10}, Sx = \frac{x}{5}, Tx = \frac{5x}{12},\ Ix = \frac{3x}{20}, Jx = \frac{x}{3}$$

Here, $ABx = A\left(\frac{4x}{10}\right) = \frac{3}{8}\left(\frac{4x}{10}\right) = \frac{3}{20}x$

$$STx = S\left(\frac{5x}{12}\right) = \frac{1}{5}\left(\frac{5x}{12}\right) = \frac{x}{12}$$

$\therefore$ $AB(X) = \left[0,\frac{3}{20}\right] \subset \left[0,\frac{1}{3}\right] = J(X)$

$$ST(X) = \left[0,\frac{1}{12}\right] \subset \left[0,\frac{3}{20}\right] = I(X)$$

or, $AB(X) \subset J(X)$ and $ST(X) \subset I(X)$

Here all the contractive condition of the Corollary (2.2) are satisfied. Hence, mappings A, B, S, T, I and J have a unique common fixed point at $x = 0$.
Now, we furnish an example to demonstrate the validity of the hypothesis of our corollary (2.3).

Example 2.7: Consider $X = [0,8]$ with the usual metric defined by
$d(x,y) = \|x-y\| = |x-y|$ and $F = R =$ Real Banach space.

Define self mappings A, B, S and T on X as

$$A0 = 0,\ \ Ax = 1,\ \ 0 < x \leq 8$$

$$B0 = 0,\quad Bx = 1,\quad 0 < x < 8,\quad B8 = 0$$
$$S0 = 0,\quad Sx = 7,\quad 0 < x < 8,\quad S8 = 4$$
$$T0 = 0,\quad Tx = 8,\quad 0 < x < 8,\quad T8 = 1$$

Here all the four maps in this example are discontinuous even at their unique common fixed point 0.
Here, $A(X) = \{0,1\} \subset T(X) = \{0,1,8\}$
And $B(X) = \{0,4\} \subset S(X) = \{0,4,7\}$

Also, the pair (A,S) and (B,T) are coincidentally commuting at $x = 0$ which is their common coincidence point.
i.e. $A0 = S0 \Rightarrow AS0 = SA0$
$B0 = T0 \Rightarrow BT0 = TB0$

By a routine calculation, we can verify that all the contractive conditions of corollary (2.3) are satisfied for $\beta_1 = \frac{1}{20}$, $\beta_2 = \frac{1}{10}$ and $\beta_3 = \frac{3}{8}$. $(2\beta_1 + \beta_2 + 2\beta_3 = 0.95 < 1)$.

4.4 CONCLUSION:

Non convex analysis, especially ordered normed spaces, normal cones and topical functions have some applications in optimization theory. In these cases, an order introduced by using vector space cones. Huang and Zhang [8] used this approach and they replaced the set of real numbers by an ordered Banach space and defined cone metric space which is generalization of metric space. In this paper, we obtain some common fixed point theorems for six mappings satisfying the different contractive conditions. Common fixed point results for weakly compatible maps which are more general than compatible mappings are obtained in the setting of cone metric spaces without requirement of the notion of continuity. Our results generalize, improve and extend the results of Goyal([3],[4]) Goyal and Gupta([5],[6])Imdad and Ali [3], Jeong and Rhoades [10] and others. In this way we can see that our result is superior to many other results.

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