BioGecko
ISSN NO: 2230-5807

Web of Science
Vol 05 Issue 01 2016

# Common Fixed Point Theorem For Compatible Mappings With The Generalized Contractive Mappings In Cone Metric Spaces

**A.K. Goyal[1*], Neelmani Gupta[2]**

[1*]Department of Mathematics,M. S. J. Govt. P.G. College, Bharatpur (Raj.)-321001,
Email: akgbpr67@rediffmail.com

[2]Department of Mathematics,Govt. Dungar College, Bikaner (Raj.)-334003,
Email: neelmanigupta04@gmail.com

***Corresponding Author:**akgbpr67@rediffmail.com

***Abstract***
***Jungck [10] introduced the concept of the more generalized commutativity, so called compatibility, which is more general than that of weak commutativity. By employing compatible mappings, we prove the following common fixed point theorem for three pairs of compatible mappings with the generalized contractive mappings in cone metric spaces. Our result extends the result of Jang et al. [7], Cho-Yoo [3] etc. in cone metric spaces.***

***Keywords :*** ***Fixed point, complete cone metric spaces, compatibility, weak commutativity,***

**2010 AMS SUBJECT CLASSIFICATION CODE : 54H25,47H10**

## 1 INTRODUCTION:

Huang and Zhang [6] generalized the concept of a cone metric space, re-placing the set of real numbers by an ordered Banach space and obtained some common fixed point theorems for mappings satisfying different contractive conditions over cone metric space. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades[2] studied common fixed point theorems in cone metric spaces. Moreover, Huang and Zhang [5], Abbas and Jungck[1], IIlic and Rakocevic [6] proved their results for normal cones. Jungck [10] generalized the concept of weak commuting by defining the term compatible mappings and proved that the weakly commuting mappings are compatible but the converse is not true. In recent years, several authors have obtained coincidence point results for various classes of mappings on a cone metric space utilizing these concepts. In this paper, we prove some common fixed point theorems for six mappings involving Ciric's type contractive condition in complete cone metric spaces. Our work generalizes some earlier results of Nesic [14], Jeong and Rhoades [8], Jang et al. [7],kang and Kim[13] and others. Some examples are also furnished to demonstrate the validity of the hypothesis.

## 2 BASIC DEFINITIONS:

The following definitions are in literature of Huang and Zhang [5].

**Definition 2.1:** Let $E$ be a real Banach space and $P$ be a subset of $E$. The subset $P$ is called a cone if and only if

(a) $P$ is closed, non-empty and $P \neq \{0\}$
(b) $a, b \in R,\ a, b \geq 0,\ x, y \in P$ implies
$ax + by \in P$
(c) $x \in P$ and $-x \in P \Rightarrow x = 0$
i.e $P \cap (-P) = \{0\}$

**Definition 2.2:** Let $P$ be a cone in a Banach space $E$ i.e. given a cone $P \subset E$, define partial ordering '$\leq$' with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$while $x \ll y$ will stand for $y - x \in Int\ P$, where $Int\ P$ denote the interior of the set $P$. This cone $P$ is called an order cone.

**Definition 2.3:** Let $E$ be a real Banach space and $P \subset E$ be an order cone. The cone $P$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|$$

The least positive number $K$ satisfying the above inequality is called the normal constant of $P$.

**Definition 2.4:** Let $X$ be a non-empty and $E$ be a real Banach space. Suppose that the mapping $d: X \times X \to E$satisfies
(d1) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$
(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$
(d3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$

Then $d$ is called a cone metric on $X$ and $(X,d)$ is called a cone metric space.
It is obvious that cone metric spaces generalize metric spaces because each metric space is a cone metric space with $E = R$ and $P = [0, +\infty[$

**Example 2.5:** (a) Let $E = R^2$, $P = \{(x,y) \in E\ |x,y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \to E$ such that
$d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant.
Then $(X,d)$ is a cone metric space.
(b) Let $E = R^n$ with $P = \{(x_i, \dots, x_n): x_i \geq 0, \forall i = 1,2,\dots,n\} X = R$ and $d: X \times X \to E$ such that

$$d(x,y) = (|x-y|, \alpha_i|x-y|, \dots, \alpha_{n-1}|x-y|)$$

where $\alpha_i \geq 0$ for all $1 \leq i \leq n-1$. Then$(X,d)$ is a cone metric space.

**Definition 2.6:** Let $(X,d)$ be a cone metric space. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$. We say that $\{x_n\}$is a
(a) convergent sequence or $\{x_n\}$ converges to $x$ if for every $c$ in $E$ with $c \gg 0$, there is an $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) \ll c$ for some fixed point $x$ in $X$where $x$ is that limit of $\{x_n\}$. This is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$. Completeness is defined in the standard way.

It was proved in [4] if $(X,d)$ be a cone metric space, $P$ be a normal cone with normal constant $K$ and $\{x_n\}$ converges to $x$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$.
(b) Cauchy sequence if for $c$ in $E$ with $c \gg 0$, there is an $n_0 \in N$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$.

It was proved in [4] if $(X,d)$ be a cone metric space, $P$ be a normal cone with normal constant $K$ and $\{x_n\}$ be a sequence in $X$, then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

**Definition 2.7:** A cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence in unique provided $P$ is a normal cone with normal constant $K$.

In recent years several definitions of conditions weaker than commutativity have appeared which facilitated significantly to extend the Jungck's theorem and several others. Foremost among of them is perhaps the weak commutativity condition introduced by Sessa [15] which can be described as follows:

**Definition 2.8:** Let $S$ and $T$ be mappings from a cone metric space $(X,d)$ into itself. Then $S$ and $T$ are said to be weakly commuting mappings on $X$ if
$d(STx, TSx) \leq d(Sx, Tx)$, for all $x \in X$.
obviously a commuting pair is weakly commuting but its converse need not be true as is evident from the following example.

**Example 2.9:** Consider the set $X = [0,1]$ with the usual metric defined by

$$d(x,y) = |x-y| = \|x-y\|$$

Define $S$ and $T: X \to X$ by

$$Sx = \frac{x}{3-2x} \text{ and } Tx = \frac{x}{3} \text{ for all } x \in X.$$

Then, we have to any $x$ in $X$

$$STx = \frac{x}{9-2x} \text{ and } TSx = \frac{x}{9-6x}$$

Hence $ST \neq TS$. Thus, $S$ and $T$ do not commute.
Again,

$$d(STx, TSx) = \left\|\frac{x}{9-2x} - \frac{x}{9-6x}\right\| = \frac{4x^2}{(9-2x)(9-6x)} \leq \frac{2x^2}{3(3-2x)} = \frac{x}{3-2x} - \frac{x}{3} = d(Sx, Tx)$$

and thus $S$ and $T$ commute weakly.

**Example 2.10:** Consider the set $X = [0,1]$ with the usual metric $d(x,y) = \|x-y\|$. Let $Sx = \frac{x}{2}$ and $Tx = \frac{x}{2+x}$, for every $x \in X$. Then, for all $x \in X$

$$STx = \frac{x}{4+2x} \text{ and } TSx = \frac{x}{4+x}$$

Hence, $ST \neq TS$. Thus, $S$ and $T$ do not commute.
Again,

$$d(STx, TSx) = \left\|\frac{x}{4+2x} - \frac{x}{4+x}\right\| = \frac{x^2}{(4+x)(4+2x)}$$

$$\leq \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x} = d(Sx, Tx)$$

and thus, $S$ and $T$ commute weakly.
Obviously, the class of weakly commuting is wider and includes commuting mappings as subclass.

Jungck [10] has observed that for $X = R$ if $Sx = x^3$ and $Tx = 2x^3$ then $S$ and $T$ are not weakly commuting. Thus it is desirable to a less restrictive concept which he termed as 'compatibility' the class of compatible mappings is still wider and includes weakly commuting mappings as subclass as is evident from the following definition of Jungck [10].

**Definition 2.11:** Let $S$ and $T$ be self mappings on a cone metric space $(X, d)$. Then $S$ and $T$ are said to be compatible mappings on $X$ if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some point $t \in X$.
Obviously, any weakly commuting pair $\{S, T\}$ is compatible, but the converse is not necessarily true, as in the following example.

**Example 2.12:** Let $Sx = x^3$ and $Tx = 2x^3$ with $X = R$ with the usual metric. Then $S$ and $T$ are compatible, since $|Tx - Sx| = |x^3| \to 0$ if and only if $|STx - TSx| = 6|x^9| \to 0$ But $|STx - TSx| \leq |Tx - Sx|$ is not true for all $x \in X$, say for example at $x = 1$.

**Definition 2.13:** Let $S$ and $T$ be self maps of a set $X$. If $w = Sx = Tx$ for some $x$ in $X$, then $x$ is called a coincidence point of $S$ and $T$ and $w$ is called a point of coincidence of $S$ and $T$.

**Remark 2.14:** Let $(X, d)$ be a cone metric space with a cone $P$. If $d(x, y) \leq hd(x, y)$ for all $x, y \in X, h \in (0,1)$, then $d(x, y) = 0$, which implies that $x = y$.

## 3MAIN RESULTS

**Theorem 3.1:**Let $(X, d)$ be a complete cone metric space and $P$ be a normal cone with normal constant K. Let A, B, S, T, I and J be self mappings from a complete cone metric space $(X, d)$ into itself satisfying the following conditions:

(i) $AB(X) \subset J(X),\ ST(X) \subset I(X)$ … (1)

(ii) $d(ABx, STy) \leq \beta_1 \max\{d(ABx, Ix), d(STy, Jy), \frac{1}{2}[d(ABx, Jy) + d(STy, Ix)], d(Ix, Jy)\}$
$+\beta_2 \max\{d(ABx, Ix), d(STy, Jy)\}$
$+\beta_3 \max\{d(ABx, Jy), d(STy, Ix)\}$ … (2)

for all $x, y \in X$ where $\beta_1, \beta_2, \beta_3 \geq 0$, $0 < \beta = \beta_1 + \beta_2 + 2\beta_3 \leq 1 (\beta_1, \beta_2, \beta_3$ are non-negative real numbers)
Suppose that

(iii) One of AB, ST, I and J is continuous. …(3)

(iv) The pairs (AB, I) and (ST,J) are compatible on X. …(4)

Then the mappings AB, ST, I and J have a unique common fixed point in X.
Furthermore, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J), (T,J) are commuting mappings then the mappings A, B, S, T, I and J have unique common fixed point.

**Proof:** Let $x_0 \in X$ be an arbitrary point. By (1), since $AB(X) \subset J(X)$, we can choose a point $x_1$ in $X$ such that $ABx_0 = Jx_1$. Also, since $ST(X) \subset I(X)$, we can fixed a point $x_2$ with $STx_1 = Ix_2$ and so on. Proceeding in the similar manner, we can define a sequence $\{z_n\}$ in $X$ such that for $n = 0,1,2,3 ...$

$$z_{2n+1} = Jx_{2n+1} = ABx_{2n},$$
$$z_{2n} = Ix_{2n} = STx_{2n-1}$$

Now, we shall show that $\{z_n\}$ is a Cauchy sequence.
Using (2), we have

$$\begin{aligned} d(z_{2n+1}, z_{2n+2}) &= d(ABx_{2n}, STx_{2n+1}) \\ &\leq \beta_1 \max\{d(ABx_{2n}, Ix_{2n}), d(STx_{2n+1}, Jx_{2n+1}), \\ &\quad \tfrac{1}{2}[d(ABx_{2n}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n})], d(Ix_{2n}, Jx_{2n+1})\} \\ &\quad +\beta_2 \max\{d(ABx_{2n}, Ix_{2n}), d(STx_{2n+1}, Jx_{2n+1})\} \\ &\quad +\beta_3\, max\{d(ABx_{2n}, Jx_{2n+1}), d(STx_{2n+1}, Ix_{2n})\} \\ &\leq \beta_1 \max\{d(z_{2n+1}, z_{2n}), d(z_{2n+2}, z_{2n+1}), \tfrac{1}{2}[d(z_{2n+1}, z_{2n+1}) \end{aligned}$$

$$+d(z_{2n+2}, z_{2n})], d(z_{2n}, z_{2n+1})\}$$
$$+\beta_2 \max\{d(z_{2n+1}, z_{2n}), d(z_{2n+2}, z_{2n+1})\}$$
$$+\beta_3\, max\{d(z_{2n+1}, z_{2n+1}), d(z_{2n+2}, z_{2n})\}$$

or, $d(z_{2n+1}, z_{2n+2}) \leq \beta_1 \max\{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{1}{2}[d(z_{2n}, z_{2n+1})$

$$+d(z_{2n+1}, z_{2n+2})], d(z_{2n}, z_{2n+1})\}$$
$$+\beta_2 \max\{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2})\}$$
$$+\beta_3\, max\{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2})\} \quad \ldots (6)$$

where $0 < \beta = \beta_1 + \beta_2 + 2\beta_3 < 1$.
In (6), if $d(z_{2n+1}, z_{2n+2}) > d(z_{2n}, z_{2n+1})$ for some positive integer n, then we get

$$d(z_{2n+1}, z_{2n+2}) \leq \beta\, d(z_{2n+1}, z_{2n+2})$$

which is a contradiction. Then, we obtain

$$d(z_{2n+1}, z_{2n+2}) \leq \beta\, d(z_{2n}, z_{2n+1})$$

Similarly, we get

$$d(z_{2n}, z_{2n+1}) = d(ABx_{2n}, STx_{2n-1})$$
$$\leq \beta_1 \max\{d(ABx_{2n}, Ix_{2n}), d(STx_{2n-1}, Jx_{2n-1}),$$
$$\frac{1}{2}[d(ABx_{2n}, Jx_{2n-1}) + d(STx_{2n-1}, Ix_{2n})], d(Ix_{2n}, Jx_{2n-1})\}$$
$$+\beta_2 \max\{d(ABx_{2n}, Ix_{2n}), d(STx_{2n-1}, Jx_{2n-1})\}$$
$$+\beta_3\, max\{d(ABx_{2n}, Jx_{2n-1}), d(STx_{2n-1}, Ix_{2n})\}$$
$$\leq \beta_1 \max\{d(z_{2n+1}, z_{2n}), d(z_{2n}, z_{2n-1}), \frac{1}{2}[d(z_{2n+1}, z_{2n-1})$$
$$+d(z_{2n}, z_{2n})], d(z_{2n}, z_{2n-1})\} + \beta_2 max\{d(z_{2n+1}, z_{2n}),$$
$$d(z_{2n}, z_{2n-1})\} + \beta_3\, max\{d(z_{2n+1}, z_{2n-1}), d(z_{2n}, z_{2n})\}$$
$$\leq \beta_1 \max\{d(z_{2n+1}, z_{2n}), d(z_{2n}, z_{2n-1}), \frac{1}{2}[d(z_{2n+1}, z_{2n})$$
$$+d(z_{2n}, z_{2n-1})], d(z_{2n}, z_{2n-1})\} + \beta_2 max\{d(z_{2n+1}, z_{2n}),$$
$$d(z_{2n}, z_{2n-1})\} + \beta_3\, max\{d(z_{2n+1}, z_{2n}), d(z_{2n}, z_{2n-1})\} \ldots (7)$$

In (7), if $d(z_{2n+1}, z_{2n}) > d(z_{2n}, z_{2n-1})$, then we get
$d(z_{2n+1}, z_{2n}) \leq \beta d(z_{2n+1}, z_{2n})$, which is a contradiction.

Thus, we get
$(z_{2n+1}, z_{2n}) \leq \beta d(z_{2n}, z_{2n-1})$ for $n = 1,2,3, \ldots$
where $0 < \beta < 1$

Now, by induction

$$d(z_{2n}, z_{2n+1}) \leq \beta\, d(z_{2n-1}, z_{2n})$$
$$\vdots$$
$$\vdots$$
$$\leq \beta^n\, d(z_0, z_1)$$

Again, for any $m > n$, we have

$$(z_n, z_m) \leq d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + \ldots + d(z_{m-1}, z_m)$$
$$\leq [\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}] d(z_1, z_0)$$
$$\leq \frac{\beta^n}{1-\beta} d(z_1, z_0)$$

Using normality of cone, we get
$||d(z_n, z_m)|| \leq \frac{\beta^n}{1-\beta} K\, ||d(z_1, z_0)||$ where $K$ is a normal constant.

This implies that $d(z_n, z_m) \to 0$ as $n, m \to \infty$ .
Hence, $\{z_n\}$defined by (5) is a Cauchy sequence. Since, $X$ is complete there exists a point $z$ in $X$ such that $\lim_{n\to\infty} z_n = z$ i.e.$\{z_n\}$ converges to some $z \in X$.

Therefore, the sequences
$z_{2n+1} = ABx_{2n} = Jx_{2n+1}$ and $z_{2n} = STx_{2n-1} = Ix_{2n}$,

which are subsequences of $\{z_n\}$ also, converges to a point $z$.
i.e. $\lim_{n\to\infty} ABx_{2n} = \lim_{n\to\infty} Jx_{2n+1} = z$ and
$\lim_{n\to\infty} STx_{2n-1} = \lim_{n\to\infty} Ix_{2n} = z$

Now, let $I$ is continuous then the sequences $\{I^2x_{2n}\}$ and $\{IBx_{2n}\}$ converge to the same point $Iz$. Since $\{AB, I\}$ are compatible on $X$, so the sequence $\{ABIx_{2n}\}$ also converge to the same point $Iz$. i.e.
$I^2x_{2n} \to Iz, ABIx_{2n} \to Iz$ as $n \to \infty$.
By (2), we get

$$d(ABIx_{2n}, STx_{2n-1}) \le \beta_1 \max \{d(ABIx_{2n}, I^2x_{2n}), d(STx_{2n-1}, Jx_{2n-1}), \frac{1}{2}[d(ABIx_{2n},, Jx_{2n-1}) + d(STx_{2n-1}, I^2x_{2n})], d(I^2x_{2n}, Jx_{2n-1})\} + \beta_2 \max \{d(ABIx_{2n}, I^2x_{2n}), d(STx_{2n-1}, Jx_{2n-1})\} + \beta_3\, max \{d(ABIx_{2n}, Jx_{2n-1}), d(STx_{2n-1}, I^2x_{2n})\}$$

Letting n$\to \infty$, we have

$$d(Iz, z) \le \beta_1 \max\{d(Iz, Iz), d(z, z), \frac{1}{2}[d(Iz, z) + d(Iz, z)], d(Iz, z)\} + \beta_2 \max \{d(Iz, Iz), d(z, z)\} + \beta_3 \max \{d(Iz, z), d(z, Iz)\} \le (\beta_1 + \beta_3)\, d(Iz, z)$$

which is a contradiction as $\beta_1 + \beta_3 < 1$, therefore
$d(Iz, z) = 0$
or, $Iz = z$.

Again, by using (2), we get

$$d(ABz, STx_{2n-1}) \le \beta_1 \max\{d(ABz, Iz), d(STx_{2n-1}, Jx_{2n-1}), \frac{1}{2}[d(ABz, Jx_{2n-1}) + d(STx_{2n-1}, Iz)], d(Iz, Jx_{2n-1})\} + \beta_2 \max \{d(ABz, Iz), d(STx_{2n-1}, Jx_{2n-1})\} + \beta_3 \max \{d(ABz, Jx_{2n-1}), d(STx_{2n-1}, Iz)\}$$

on letting n$\to \infty$, we get

$$d(ABz, z) \le \beta_1 \max\{d(ABz, z), d(z, z), \frac{1}{2}[d(ABz, z) + d(z, z)], d(z, z)\} + \beta_2 \max \{d(ABz, z), d(z, z)\} + \beta_3 \max \{d(ABz, z), d(z, z)\} \le (\beta_1 + \beta_2 + \beta_3)\, d(ABz, z)$$

which is a contradiction as $\beta_1 + \beta_2 + \beta_3 < 1$. Therefore, we get $ABz = z$ .

Since $AB(X) \subset J(X)$ and $z$ is in the range of AB i.e. $z \in AB(X)$. Therefore, there exists a point $z' \in X$ such that $z = ABz = Jz'$.
Now,

$$d(z, STz') = d(ABz, STz') \le \beta_1 \max\{d(ABz, Iz), d(STz', Jz'), \frac{1}{2}[d(ABz, Jz')$$

$$+d(STz', Iz)], d(Iz, Jz')\} + \beta_2 \max\{d(ABz, Iz), d(STz', Jz')\}$$
$$+\beta_3 \max\{d(ABz, Jz'), d(STz', Iz)\}$$
$$\leq \beta_1 \max\{d(z,z), d(STz',z), \tfrac{1}{2}[d(z,z) + d(STz',z)], d(z,z)\}$$
$$+\beta_2 \max\{d(z,z), d(STz',z)\} + \beta_3 \max\{d(z,z), d(STz',z)\}$$

or, $d(z, STz') \leq (\beta_1 + \beta_2 + \beta_3)\, d(STz', z)$ which is a contradiction as $\beta_1 + \beta_2 + \beta_3 < 1$, which gives $z = STz'$. Therefore, $z = STz' = Jz'$, which shows that $z'$ is a coincidence point of ST and J.

Since, ST and J are compatible on $X$ and $Jz' = STz' = z$. Therefore, we have $d(JSTz', STJz') = 0$.
Hence, $Jz = JSTz' = STJz' = STz$
or, $Jz = STz$.
By (2), we get

$$d(z, Jz) = d(ABz, STz)$$
$$\leq \beta_1 \max\{d(ABz, Iz), d(STz, Jz), \tfrac{1}{2}[d(ABz, Jz) + d(STz, Iz)], d(Iz, Jz)\}$$
$$+\beta_2 \max\{d(ABz, Iz), d(STz, Jz)\} + \beta_3 \max\{d(ABz, Jz), d(STz, Iz)\}$$
$$\leq \beta_1 \max\{d(z,z), d(Jz, Jz), \tfrac{1}{2}[d(z, Jz) + d(z, Jz)], d(z, Jz)\}$$
$$+\beta_2 \max\{d(z,z), d(Jz, Jz)\} + \beta_3 \max\{d(z, Jz), d(Jz, z)\}$$

or, $d(z, Jz) \leq (\beta_1 + \beta_3)\, d(z, Jz)$ a contradiction so that $z = Jz = STz$, which shows that $z$ is a common fixed point of AB, ST, I and J.

Now, suppose that $AB$ is continuous so that the sequences $\{AB^2x_{2n}\}$ and $\{ABIx_{2n}\}$ converge to $ABz$. Since, $\{AB, I\}$ are compatible on $X$, it follows that $\{IABx_{2n}\}$ also converges to $ABz$ i.e.
$AB^2x_{2n} \to ABz, IABx_{2n} \to ABz$ as $n \to \infty$.
By (2), we have

$$d(AB^2x_{2n}, STx_{2n-1}) \leq \beta_1 \max\{d(AB^2x_{2n}, IABx_{2n}), d(STx_{2n-1}, Jx_{2n-1}),$$
$$\tfrac{1}{2}[d(AB^2x_{2n}, Jx_{2n-1}) + d(STx_{2n-1}, IABx_{2n})],$$
$$d(IABx_{2n}, Jx_{2n-1})\}$$
$$+\beta_2 \max\{d(AB^2x_{2n}, IABx_{2n}), d(STx_{2n-1}, Jx_{2n-1})\}$$
$$+\beta_3 \max\{d(AB^2x_{2n}, Jx_{2n-1}), d(STx_{2n-1}, IABx_{2n})\}$$

which on letting $n \to \infty$, reduces to

$$d(ABz, z) \leq \beta_1 \max\{d(ABz, ABz), d(z,z), \frac{1}{2}[d(ABz, z) + d(ABz, z)], d(ABz, z)\}$$
$$+\beta_2 \max\{d(ABz, ABz), d(z,z)\} + \beta_3 \max\{d(ABz, z), d(z, ABz)\}$$
$$\leq (\beta_1 + \beta_3) d(ABz, z)$$

which is a contradiction, yielding thereby
$ABz = z$ as $\beta_1 + \beta_3 < 1$.

Since $z$ is in the range of $AB$ and $AB(X) \subset J(X)$, there always exists a point $z'$ such that $Jz' = z = ABz$. Then

$$d(AB^2x_{2n}, STz') \leq \beta_1 \max\{d(AB^2x_{2n}, IABx_{2n}), d(STz', Jz'), \frac{1}{2}[d(AB^2x_{2n}, Jz')$$
$$+d(STz', IABx_{2n})], d(IABx_{2n}, Jz')\}$$
$$+\beta_2 \max\{d(AB^2x_{2n}, IABx_{2n}), d(STz', Jz')\}$$
$$+\beta_3 \max\{d(AB^2x_{2n}, Jz'), d(STz', IABx_{2n})\}$$

which on letting $n \to \infty$ reduces to

$$d(z, STz') \leq \beta_1 \max\{d(z,z), d(STz', z), \frac{1}{2}[d(z,z) + d(STz', z)], d(z,z)\}$$
$$+\beta_2 \max\{d(z,z), d(STz', z)\} + \beta_3 \max\{d(z,z), d(STz', z)\}$$

$\leq (\beta_1 + \beta_2 + \beta_3) d(STz', z)$, a contradiction which yields
$z = STz' = Jz'$.

Thus, the pair $(ST, J)$ has a coincidence point $z'$.
Since, the pair $(ST, J)$ is compatible on $X$ and $Jz' = STz' = z$, we have $d(JSTz', STJz') = 0$ [by def.of compatible].
Hence, $Jz = J(STz') = ST(Jz') = STz$, which shows that $STz = Jz$.
Further, by (2), we have $d(ABx_{2n}, STz) \leq \beta_1 \, max\{d(ABx_{2n}, Ix_{2n}), d(STz, Jz), \frac{1}{2}[d(ABx_{2n}, Jz)$
$+d(STz, Ix_{2n})], d(Ix_{2n}, Jz)\}$
$+\beta_2 \, max\{d(ABx_{2n}, Ix_{2n}), d(STz, Jz)\}$
$+\beta_3 \, max\{d(ABx_{2n}, Jz), d(STz, Ix_{2n})\}$

which on letting $n \rightarrow \infty$, we get

$$d(z, STz) \leq \beta_1 \max\{d(z,z), d(STz, STz), \frac{1}{2}[d(z, STz) + d(STz, z)], d(z, STz)\}$$
$$+\beta_2 \max \{d(z,z), d(STz, STz)\} + \beta_3 \max \{d(z, STz), d(STz, z)\}$$
$$\leq (\beta_1 + \beta_3) \, d(z, STz), \text{ a contradiction}$$

which implies that $STz = z = Jz$ as $\beta_1 + \beta_3 < 1$
Since, $ST(X) \subset I(X)$ and $STz = z$, then there exists a point $z''$ in $X$ such that $Iz'' = z$. Thus,
$d(ABz'', z) = d(ABz'', STz)$

$$\leq \beta_1 \, max\{d(ABz'', Iz''), d(STz, Jz), \frac{1}{2}[d(ABz'', Jz)$$
$$+d(STz, Iz'')], d(Iz'', Jz)\}$$
$$+\beta_2 \max \{d(ABz'', Iz''), d(STz, Jz)\}$$
$$+\beta_3 \max \{d(ABz'', Jz), d(STz, Iz'')\}$$
$$\leq \beta_1 \max\{d(ABz'', z), d(z,z), \frac{1}{2}[d(ABz'', z) + d(z,z)], d(z,z)\}$$
$$+\beta_2 \max \{d(ABz'', z), d(z,z)\} + \beta_3 \max \{d(ABz'', z), d(z,z)\}$$

or, $d(ABz'', z) \leq (\beta_1 + \beta_2 + \beta_3) \, d(ABz'', z)$, a contradiction which implies that
$ABz'' = z$ as $\beta_1 + \beta_2 + \beta_3 < 1$.

Again, since $(AB, I)$ are compatible on $X$ and $ABz'' = Iz'' = z$, we have
$d(IABz'', ABIz'') = 0$.

Therefore, $Iz = I(ABz'') = AB(Iz'') = ABz$.

Hence, $ABz = Iz = z$. Thus, we have proved that $z$ is a common fixed point of AB, ST, I and J.

Instead of AB or I, if the mappings ST or J is continuous, then the proof that $z$ is a common fixed point of AB, ST, I and J is similar.

To show that $z$ is unique, let u be another fixed point of AB, ST, I and J. Then
$d(z, u) = d(ABz, STu)$

$$\leq \beta_1 \max\{d(ABz, Iz), d(STu, Ju), \frac{1}{2}[d(ABz, Ju) + d(STu, Iz)], d(Iz, Ju)\}$$
$$+\beta_2 \max \{d(ABz, Iz), d(STu, Ju)\} + \beta_3 \max \{d(ABz, Ju), d(STu, Iz)\}$$
$$\leq \beta_1 \max\{d(z,z), d(u,u), \frac{1}{2}[d(z,u) + d(u,z)], d(z,u)\}$$
$$+\beta_2 \max \{d(z,z), d(u,u)\} + \beta_3 \max \{d(z,u), d(u,z)\}$$

or, $d(z, u) \leq (\beta_1 + \beta_3) \, d(z, u)$, a contradiction yielding thereby
$z = u$ as $\beta_1 + \beta_3 < 1$.

Finally, we will prove that $z$ is also a common fixed point of A, B, S, T, I and J. Let both the pairs (AB,I) and (ST,J) have a unique common fixed point $z$. Then

Az = A(ABz) = A(BAz) = AB(Az)
Az = A(Iz) = I(Az)
Bz = B(ABz) = B(A(Bz)) = BA(Bz)= AB(Bz)
Bz = B(Iz) = I(Bz)
which implies that (AB,I) has common fixed points which are Az and Bz. We get, thereby Az = z = Bz = Jz = ABz, by virtue of uniqueness of common fixed point of pair (AB, I).

Similarly, using the commutativity of (S,T), (S,J) and (T, J), Sz = z = Tz = Jz = STz can be shown.

Now, we claim that Az = Sz (Bz = Tz), a common fixed point of both the pairs (AB,I) and (ST,J). We have

d(Az, Sz) = d(A(ABz), S(STz))
= d(A(BAz), S(TSz))
= d(AB(Az), ST(Sz))

$$\leq \beta_1 \max\{d(AB(Az), I(Az)), d(ST(Sz), J(Sz)), \tfrac{1}{2}[d(AB(Az), J(Sz)) + d(ST(Sz), J(Az))], d(I(Az), J(Sz))\}$$
$$+\beta_2 \max\ \{d(AB(Az), I(Az)), d(ST(Sz), J(Sz))\}$$
$$+\beta_3 \max\ \{d(AB(Az), J(Sz)), d(ST(Sz), I(Az))\}$$
$$\leq \beta_1 \max\{d(Az, Az), d(Sz, Sz), \tfrac{1}{2}[d(Az, Sz) + d(Sz, Az)], d(Az, Sz)\}$$
$$+\beta_2 \max\ \{d(Az, Az), d(Sz, Sz)\} + \beta_3 \max\ \{d(Az, Sz), d(Sz, Az)\}$$

or, $d(Az, Sz) \leq (\beta_1 + \beta_3)\ d(Az, Sz)$, a contradiction which implies that $Az = Sz$.
Similarly, Bz = Tz can be shown. Thus, z is the unique common fixed point of A, B, S, T, I and J.
The following corollary follows immediately from our theorem (3.1).

**Corollary 3.2:** Let $(X, d)$ be a complete cone metric space and $P$ be a normal cone with normal constant K. Let A, B, S, T, I and J be self mappings from a complete cone metric space $(X, d)$ into itself satisfying the following conditions:

(i) $AB(X) \subset J(X),\ ST(X) \subset I(X)$

(ii) $d(ABx, STy) \leq \beta_1 \max\{d(ABx, Ix), d(STy, Jy), \tfrac{1}{2} d(ABx, Jy), \tfrac{1}{2} d(STy, Ix), d(Ix, Jy)\} + \beta_2\, max\,\{d(ABx, Ix), d(STy, Jy)\} + \beta_3 \max\{d(ABx, Jy), d(STy, Ix)\}$

for all $x, y \in X$ where $\beta_1, \beta_2, \beta_3 \geq 0, \beta_1 + \beta_2 + 2\beta_3 \leq 1 (\beta_1, \beta_2, \beta_3$ are non-negative real numbers)
Suppose that

(iii) One of AB, ST, I and J is continuous.

(iv) The pairs (AB, I) and (ST,J) are compatible on X.

Then the mappings AB, ST, I and J have a unique common fixed point in X.

Furthermore, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J), (T,J) are commuting mappings then the mappings A, B, S, T, I and J have unique common fixed point.
If we put AB = A, ST = B in theorem (3.1), we get the following, which generalize the result of Jang et al. [7] in cone metric spaces.

**Corollary 3.3:** Let $(X, d)$ be a complete cone metric space and $P$ be a normal cone with normal constant $K$. Let A, B, S and T be self mappings from a complete cone metric space $(X, d)$ into itself satisfying the conditions:

(i) $A(X) \subset T(X), B(X) \subset S(X)$

(ii) $d(Ax, By) \leq \beta_1 \max\{d(Ax, Sx), d(By, Ty), \tfrac{1}{2}[d(Ax, Ty) + d(By, Sx)], d(Sx, Ty)\} + \beta_2 \max\ \{d(Ax, Sx), d(By, Ty)\} + \beta_3 \max\{d(Ax, Ty), d(By, Sx)\}$

For all $x, y \in$X, where $0 < \beta = \beta_1 + \beta_2 + 2\beta_3 < 1$ ($\beta_1, \beta_2, \beta_3$ are non-negative real numbers).

(iii) Suppose that one of A, B, S and T is continuous.

(iv) The pairs $(A, S)$and $(B, T)$ are compatible on X then A, B, S and T have a unique common fixed point in X.

Taking AB=A, ST=B, $\beta_2 = 0$ in theorem (3.1), we obtain the following, which generalize the result of Cho-Yoo [3] in cone metric spaces.

**Corollary 3.4:** Let $(X, d)$ be a complete cone metric space and $P$ be a normal cone with normal constant $K$. Let A, B, S and T be mappings from a complete cone metric space $(X, d)$ into itself satisfying the conditions:

(i) $A(X) \subset T(X), B(X) \subset S(X)$

(ii) $d(Ax, By) \leq \beta_1 \max\{d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(Ax, Ty) + d(By, Sx)], d(Sx, Ty)\}$
$+\beta_3 \max\{d(Ax, Ty), d(By, Sx)\}$

For all $x, y \in$X, where $0 < \beta = \beta_1 + 2\beta_3 < 1$ $(\beta_1, \beta_3$ are non-negative real numbers).

(iii) Suppose that one of A, B, S and T is continuous.

(iv) The pairs $(A, S)$and $(B, T)$ are compatible on X then A, B, S and T have a unique common fixed point in X.

If we put AB=A, ST=B, $\beta_2 = 0$ and $\beta_3 = 0$ in theorem (3.1), we obtain the following, which improve and generalize the result of Kang-Kim [13] and Jungck [11] in cone metric spaces.

**Corollary 3.5:** Let $(X, d)$ be a complete cone metric space and $P$ be a normal cone with normal constant $K$. Let A, B, S and T be mappings from a complete cone metric space $(X, d)$ into itself satisfying the conditions:

(i) $A(X) \subset T(X), B(X) \subset S(X)$

(ii) $d(Ax, By) \leq \beta_1 \max\{d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(Ax, Ty) + d(By, Sx)], d(Sx, Ty)\}$

For all $x, y \in$X, where $0 < \beta_1 < 1$ ($\beta_1$ is non-negative real number).

(iii) Suppose that one of A, B, S and T is continuous.

(iv) The pairs $(A, S)$and $(B, T)$ are compatible on X then A, B, S and T have a unique common fixed point in X.

**Example 3.6:** Consider $X = [0,1]$ with the usual metric defined by
$d(x, y) = \|x - y\| = |x - y|$ and $F = R =$ Real Banach space.

Define self mappings A, B, S, T, I and J by
$Ax = \frac{2x}{3}$, $Bx = \frac{3x}{4}$, $Sx = \frac{x}{4}$, $Tx = \frac{4x}{5}$, $Ix = \frac{x}{4}$ and $Jx = \frac{3x}{4}$ for all $x \in X$, respectively.

Then, all the hypothesis oftheorem (3.1) are satisfied for
$\beta_1 = \frac{1}{5}$, $\beta_2 = \frac{1}{3}$ and $\beta_3 = \frac{1}{20}$ where $\beta_1 + \beta_2 + 2\beta_3 < 1$. Hence, 0 is a unique common fixed point of A, B, S, T, I and J.

Now, we give some examples to illustrate our corollary (3.3).

**Example 3.7:** Let $X = [0, \infty)$ with the usual metric defined by $d(x, y) = \|x - y\| = |x - y|$ and $E = R =$ Real Benach space.
Define self mappings A, B, S, T : $X \to X$ by
$Ax = Bx = \frac{1}{8}x + 1$, $Sx = Tx = \frac{1}{2}x + 1$ for all $x \in X$, respectively.
Now, we get

$$d(Ax, Ay) = \frac{1}{4}d(Sx, Sy)$$
$$\leq \frac{1}{4} max\{d(Ax, Sx), d(Ay, Sy), \frac{1}{2}[d(Ax, Sy) + d(Ay, Sx)], d(Sx, Sy)\}$$

$+\beta_2 max\{d(Ax,Sx),d(Ay,Sy)\}+\beta_3 max\{d(Ax,Sy),d(Ay,Sx)\}$

For all $x,y \in X$, where $0 \le \beta_2+2\beta_3 < \frac{3}{4}$. Here, all the conditions of the corollary (3.3) are satisfied except the condition of compatibility of the pair $(A,S)$. Therefore, $A$ and $S$ don't have a common fixed point in $X$.

**Example 3.8:** Let $X=[0,1]$ with the usual metric defined by

$d(x,y)=\|x-y\|=|x-y|$ and $E=R=$ Real Benach space.

Define A, B, S and T: $X \to X$ by

$$Ax=0,\ Bx=\begin{cases}\frac{1}{4}\ if\ x=\frac{1}{2}\\ \frac{1}{4}x\ if\ x\neq\frac{1}{2}\end{cases},\ Sx=x,\ \ Tx=\begin{cases}1\ if\ x=\frac{1}{2}\\ x\ if\ x\neq\frac{1}{2}\end{cases}$$

for all $x \in X$ respectively. We get

$$d(Ax,By)=\begin{cases}\frac{1}{4}=\frac{1}{3}d(By,Ty)\ if\ y=\frac{1}{2}\\ \frac{1}{4}y=\frac{1}{3}d(By,Ty)\ if\ y\neq\frac{1}{2}\end{cases}$$

$$\le \frac{1}{3}max\{d(Ax,Sx),d(By,Ty),\tfrac{1}{2}[d(Ax,Ty)+d(By,Sx)],d(Sx,Ty)\}$$

$$+q\ max\{d(Ax,Sx),d(By,Ty)\}+r\ max\{d(Ax,Ty),d(By,Sx)\}$$

For all $x,y \in X$, where $0 \le \beta_2+2\beta_3 < \frac{2}{3}$.

Therefore, all the conditions of corollary (3.3) are satisfied. Consequently, 0 is a unique common fixed point of A, B, S and T.

**REFERENCES**


1. Abbas, M. and Jungck, G., "Common fixed point results for noncommitting mappings without continuity in cone metric spaces", J. Math. Anal. Appl., 341(1), 416–420, 2008.
2. Abbas, M. and Rhoades, B.E., "Fixed and periodic results in cone metric spaces", Applied Mathematics Letters, 22(4), 511–515, 2009.
3. Cho, S.Y. and Yoo, M.J., "Common fixed point of generalized contractive mappings", East Asian Math. J., 25, 1-10, 2009.
4. Han, Y. and Xu, S., "Some fixed point theorems for expanding mappings without continuity in cone metric spaces", Fixed Point Theory Appl., Vol. 03, 1–9, 2013.
5. Huang, G. and Zhang, X., "Cone metric spaces and fixed point theorems of contractive mappings", J. Math. Anal. Appl., 332,1468–1476, 2007.
6. Ilic, D. and Rakocevic, V., "Common fixed points for maps on cone metric spaces", J. Math. Anal. Appl., 341(2),876–882, 2008.
7. Jang, J.K., Hun, J.K., Bae, N.J., Kim, J.H., Lee, P.M. and Kang, S.M., "Common fixed point theorems of compatible mappings in metric spaces", Int. J. Pure and Applied Maths., 84(1), 171-183, 2013.
8. G.S. Jeong and B.E. Rhoades., "Some remarks for improving fixed point theorem for more than two maps", Ind. J. Pure. Appl. Math ***28 (9) (1997)***, 1177-1196.
9. G. Jungck., "Commuting mappings and fixed points", Amer. Math. Monthly., ***83 (1976)***, 261-263.
10. G. Jungck., "Compatible mappings and common fixed point", Internet. J. Math and Math. Sci., ***9 (4) (1986)***, 771-779.
11. Jungck, G., "Compatible mappings and common fixed points", Internat. J. Math. & Math. Sci., 11(2), 285-288, 1988.
12. G. Jungck and B.E. Rhoades., "Fixed point for set valued function without continuity", Ind. J. Pure. Appl. Math., ***29 (3) (1998)***, 227-238.
13. Kang, S.M. and Kim, Y.R., "Common fixed point theorem", Math. Japon., 37, 1031-1039, 1992.
14. Slobodan, C. Nesic., "Results on fixed points of aymptotically regular mappings", Ind. J. Pure. Appl. Math., ***30 (5) (1999),*** 491-494.
15. S. Sessa., "On a weak commutativity condition in fixed point consideration", Publ. Inset. Math., ***32 (1982)***, 149-153.