ABSOLUTELY HARMONIOUS LABELING OF SOME DERIVED GRAPHS

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Abstract
Absolutely harmonious labeling f is an injection from the vertex set of a graph G with q edges to the set $\{0,1,2,\ldots,q-1\}$, if each edge uv is assigned $f(u)+f(v)$ then the resulting edge labels can be arranged as $\{a_0,a_1,a_2,\ldots,a_{q-1}\}$ where $a_i=q-i$ or $q+i, 0\leq i\leq q-1$. However, when G is a tree one of the vertex labels may be assigned to exactly two vertices. A graph which admits Absolutely harmonious labeling is called Absolutely Harmonious Graph. In this paper, we study absolutely harmonious labeling of some derived graph.

Keywords: Jelly fish, Star related graph, Butterfly graph, Fire cracker

1. Introduction

In this paper, we consider finite and undirected graphs. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A vertex labeling of a graph G is an assignment f of labels to the vertices that induces a label for each edge xy depending on the vertex labels. M.seenivasan and A.Lourdusamy [3] introduced another variation of harmonious labeling, namely, Absolutely harmonious labeling of graphs. In this paper we study the absolutely harmonious labeling of some derived graphs.

Definition 1.1.

Absolutely harmonious labeling f is an injection from the vertex set of a graph G with q edges to the set $\{0,1,2,\ldots,q-1\}$, if each edge uv is assigned $f(u)+f(v)$ then the resulting edge labels can be arranged as $\{a_0, a_1, a_2, \ldots, a_{q-1}\}$ where $a_i = q - i$ or $q + i, 0 \leq i \leq q - 1$. A graph which admits absolutely harmonious labeling is called Absolutely Harmonious Graph.

Theorem 1.1.

P_n^2 is an Absolutely Harmonious Graph.

Proof.

Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n^2
and $E(P_n^2) = \{v_i\, v_{i+1}: 1 \leq i \leq n-1\} \cup \{v_i v_{i+2} : 1 \leq i \leq n-2\}$
Here, P_n^2 is of order n and size $2n - 3$.
Now,Define $f: V(P_n^2) \to \{0,1,2,3, \ldots, , q-1\}$ as follows
$f(v_i) = i - 1, 1 \leq i \leq n$
The induced edge label are as follows
$f^*(v_i v_{i+1}) = a_{2K}$; $1 \leq i \leq n-1; n-2 \leq k \leq 0$
$f^*(v_i v_{i+2}) = a_{2k-1}$; $1 \leq i \leq n-2; n-2 \leq k \leq 1$
From the above, $a_0, a_1, a_2, \ldots, a_{q-1}$ where $a_i = q - i$ or $q + i, 0 \leq i \leq q-1$ are the arranged edge labels.
Therefore f admits absolutely harmonious labeling of P_n^2 and Hence P_n^2 is an Absolutely Harmonious Graph.

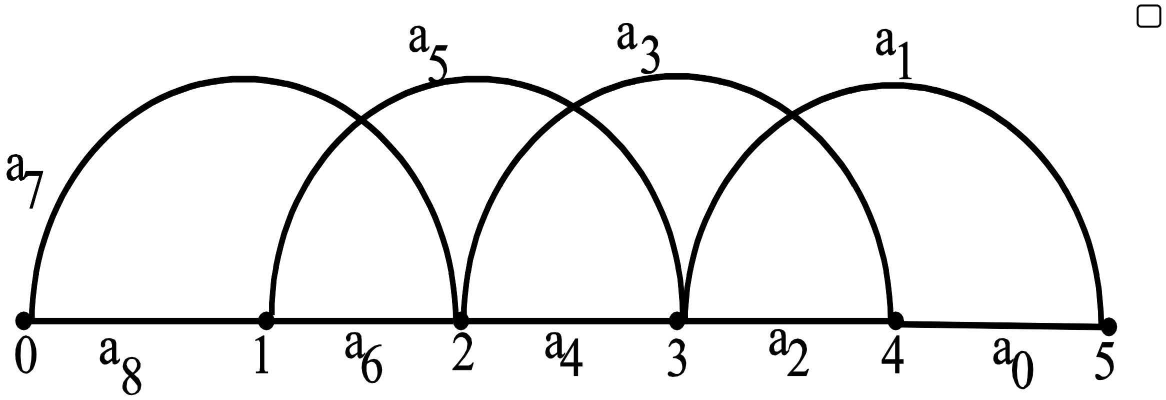

Figure 1: P_6^2

Theorem 1.2.

Jelly fish $J(n, n)$ is an Absolutely Harmonious Graph.

Proof.

Let $G = J(n, n)$.
The vertex set and the edge set of G are given by
$V(G) = \{(u, v, x, y), (u_i v_i, 1 \leq i \leq n)\}$
and $E(G) = \{[(ux) \cup (uy) \cup (vx) \cup (vy) \cup (xy)] \cup [(uu_i; 1 \leq i \leq n] \cup [(vv_i; 1 \leq i \leq n]\}$
Here,G is of order $2n + 4$ and size $2n + 5$
Now,Define $f: V(G) \to \{0,1,2,3, \ldots, , q-1\}$ as follows
$f(u) = 1$
$f(v) = 2$
$f(x) = 3$
$f(y) = 0$
$f(u_i) = q - i, 1 \leq i \leq n$
$f(v_i) = 4 + i, 1 \leq i \leq n-1$
Then the induced edge labels are as follows
$f^*(uy) = a_{q-1}$
$f^*(yv) = a_{q-2}$

$f^*(xy) = a_{q-3}$
$f^*(xu) = a_{q-4}$
$f^*(vx) = a_{q-5}$
$f^*(uu_i) = a_k\ ; 1 \le i \le n; 0 \le k \le n-1$
$f^*(vv_i) = a_{n+k};\ n \le i \le 1; 0 \le k \le n-1$
From the above, $a_0, a_1, a_2, \dots, a_{q-1}$ where $a_i = q - i$ or $q + i, 0 \le i \le q-1$ are the arranged edge labels.
Therefore, f is an absolutely harmonious labeling of the Jelly fish $J(n, n)$
and Hence the Jelly fish $J(n, n)$ is an Absolutely Harmonious Graph.

Definition 1.2. ▢
Let $S_{m,3}$ be a star graph with $3m + 1$ vertices and $3m$ edges.
Let $V = \{u\} \cup \{x_i: 1 \le i \le m\} \cup \{y_i: 1 \le i \le m\} \cup \{z_i: 1 \le i \le m\}$
be the vertex set of star graph where u is a center vertex x_i, y_i, v_i are the vertices of the path P_3 for $1 \le i \le m$. and $E = \{ux_i: 1 \le i \le m\} \cup \{x_iy_i: 1 \le i \le m\} \cup \{y_iz_i: 1 \le i \le m\}$ be the edge set of the star graph $S_{m,3}$.It is denoted as in the below figure.

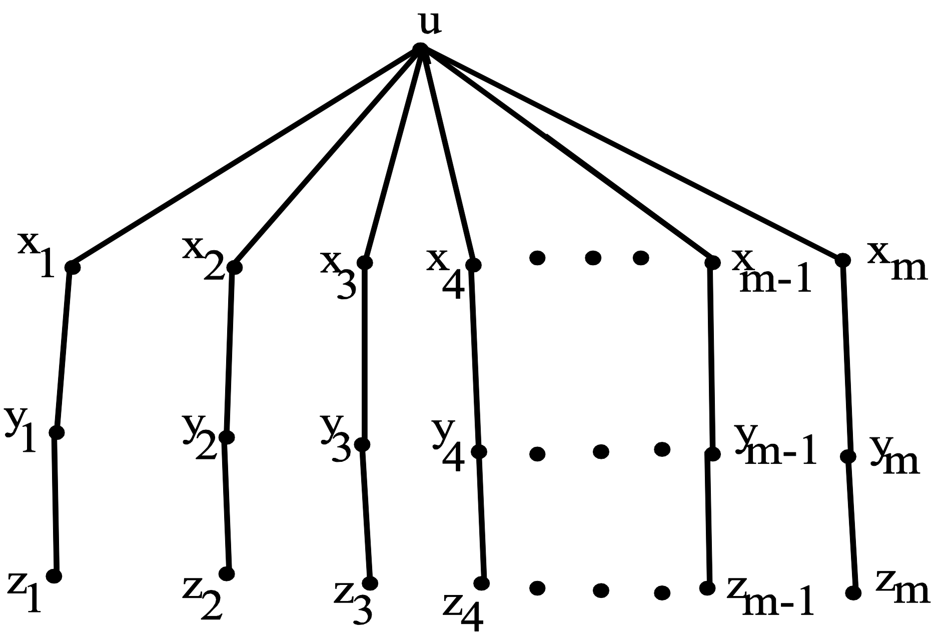

Figure 2: $S_{m,3}$

Theorem 1.3.
The Star graph $S_{m,3}$ is Absolutely Harmonious.
Proof.
Let G be a Star graph $S_{m,3}$ with $3m + 1$ vertices and $3m$ edges.
Now,Define $f: V(G) \to \{0,1,2,3, \dots, , q-1\}$ as follows
$f(u) = 0$
$f(x_i) = i, 1 \le i \le m$
$f(y_i) = m + i, 1 \le i \le m$
$f(z_i) = (2m - 1) + i, 3 \le i \le m$
The induced edge labels are as follows
$f^*(uu_i) = a_{q-i}; 1 \le i \le m$
$f^*(x_iy_i) = a_{2j}; 1 \le i \le m; m - 1 \le j \le 0$
$f^*(y_iz_i) = a_{2i-1}; 1 \le i \le m$
From the above, $a_0, a_1, a_2, \dots, a_{q-1}$

where $a_i = q - i$ or $q + i, 0 \leq i \leq q - 1$are the arranged edge labels
Therefore f is an absolutely harmonious labeling of the Star graph $S_{m,3}$
and Hence the Star graph $S_{m,3}$is an Absolutely Harmonious Graph.

Theorem 1.4. ▢
$K_{1,n,n}$ is an Absolutely Harmonious Graph.
Proof.
Let $G = K_{1,n,n}$
The vertex set and the edge set of G are given by
$V(G) = \{u, v, w_i, 1 \leq i \leq n\}$
and $E(G) = \{[(uv)] \cup [(uw_i); 1 \leq i \leq n] \cup [(vw_i); 1 \leq i \leq n]\}$
Here,G is of order $n + 2$ and size $2n + 1$
Now,Define $f: V(G) \to \{0,1,2,3, \ldots, q - 1\}$ as follows
$f(u) = n + 1, f(v) = 0$
$f(w_i) = i; 1 \leq i \leq n$
The induced edge labels are as follows
$f^*(vw_i) = a_{q-i} 1 \leq i \leq n$
$f^*(uw_i) = a_{[(q-n)+i]}; 1 \leq i \leq n$
$f^{*(uv)} = a_{[q-(n+1]}$
From the above, $a_0, a_1, a_2, \ldots, a_{q-1}$ where $a_i = q - i$ or $q + i, 0 \leq i \leq q - 1$are the arranged edge labels.
Therefore f admits absolutely harmonious labeling of $K_{1,n,n}$
and Hence $K_{1,n,n}$ is an Absolutely Harmonious Graph.

Definition 1.3. ▢
Let G_1 and G_2 be two copies of a graph. We construct a new graph $G' = < G_1 \Delta G_2 >$ which is obtained by joining the apex vertices of G_1 and G_2 by an edge as well as to a new vertex v'.
Theorem 1.5.
$< K_{1,n}^{(1)} \Delta K_{1,n}^{(2)} >$ is an Absolutely Harmonious Graph.
Proof.
Let $G = < K_{1,n}^{(1)} \Delta K_{1,n}^{(2)} >$.
Let $v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}$ be the pendant vertices of $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}$be the pendant vertices of $K_{1,n}^{(2)}$
Now,u and v are the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively and u, v are adjacent to a new common vertex w.
Here, G is of order $2n + 3$ and size $2n + 3$.
Now,Define $f: V(G) \to \{0,1,2,3, \ldots, q - 1\}$ as follows
$f(w) = n + 2$
$f(u) = 0$
$f(v) = n + 1$
$f(u_i) = i, 1 \leq i \leq n$
$f(v_j) = n + 2 + j, 1 \leq j \leq n$
Then the induced edge labels are as follows
$f^*(uw) = a_{n+1}$
$f^*(uv) = a_{n+2}$
$f^*(vw) = a_0$
$f^*(uu_k) = a_{q-i}; 1 \leq i \leq n; 1 \leq k \leq n$
$f^*(vv_k) = a_k ; 1 \leq k \leq n$

From the above, $a_0, a_1, a_2, \dots, a_{q-1}$ where $a_i = q - i$ or $q + i, 0 \leq i \leq q - 1$ are the arranged edge labels.
Therefore, f admits absolutely harmonious labeling.
and Hence, $G =< K_{1,n}^{(1)} \Delta \ K_{1,n}^{(2)}$ >is an Absolutely Harmonious Graph.

□

Definition 1.4.
The Butterfly graph $B_{n,m}$ where n, m are positive integers is defined as the two cycles of the same order n sharing a common vertex with an arbitrary number of m pendant edges are attached at a common vertex vertex.
Theorem 1.6.
The Butterfly graph $B_{3,m}$, $m \geq 2$ is an Absolutely Harmonious Graph.
Proof.
Let $G = B_{3,m}$, be a Butterfly graph.
Let $u_1, u_2, u_3, u_4, u_5, w_1, w_2, w_3, \dots, w_m$ be the vertices of the two cycle C_3 and u_3 be the center vertex of the two cycles.
Let $w_1, w_2, w_3, \dots, w_m$ be the adjacent vertices of u_3 and the Edge set is $\{(u_3, w_i), (u_i, u_{i+1}), (u_1, u_3), (u_3, u_5)\}$.
Here,$G = B_{3,m}$ is of order $2n + m - 1$ and size $2n + m$.
Now,Define $f: V(G) \rightarrow \{0,1,2,3, \dots,, q - 1\}$ as follows
Case 1: m=2
$f(u_1) = 2$
$f(u_2) = q - 2$
$f(u_3) = 0$
$f(u_4) = 1$
$f(u_5) = 3$
$f(w_1) = q - 1$
$f(w_2) = q - 3$
Case 2: $m > 2$
$f(u_1) = 2$
$f(u_2) = 1$
$f(u_3) = 0$
$f(u_4) = 4$
$f(u_5) = m + 2$
Now,the label of $f(w_i)$ for $1 \leq i \leq m$ is as follows
Case 3: m=3
$f(w_j) = m + j + 3, 1 \leq i \leq m, 0 \leq j \leq m$
Case 4: $m = 4$
$f(w_1) = 5$
$f(w_2) = 6$
$f(w_3) = 8$
$f(w_4) = 9$
Case 5: $m > 4$
$f(w_1) = 5$
$f(w_i) = k + 5, 1 \leq k \leq m - 4, 2 \leq i \leq m - 3$
$f(w_i) = m + t + 3, 0 \leq t \leq 2, m - 2 \leq i \leq m$
It can be easily verifed that $a_0, a_1, a_2, \dots, a_{q-1}$
where $a_i = q - i$ or $q + i, 0 \leq i \leq q - 1$ are the arranged edge labels.
Therefore, f is an absolutely harmonious labeling of the Butterfly graph $B_{3,m}, m \geq 2$
and Hence the Butterfly graph $B_{3,m}, m \geq 2$ is an Absolutely Harmonious Graph.

□

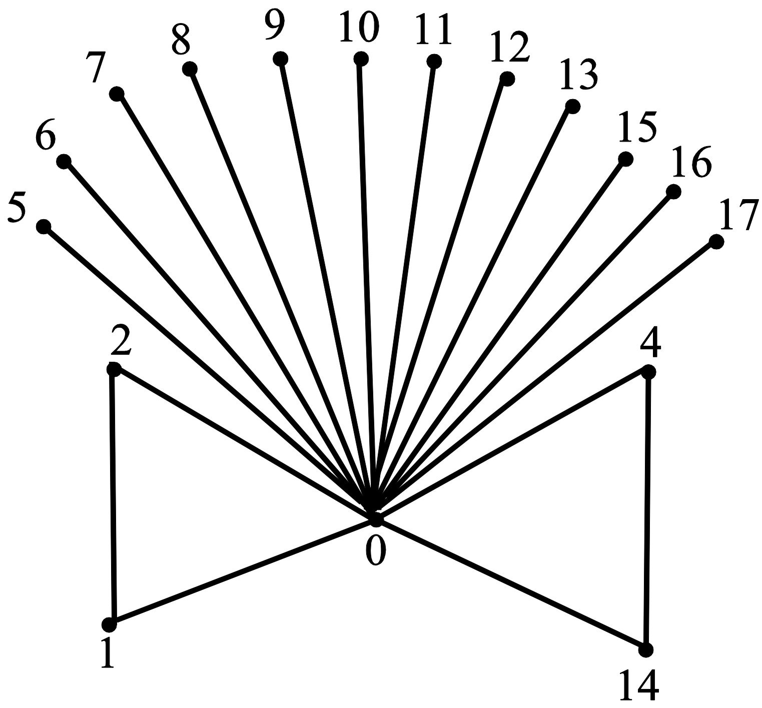

Figure 3: $B_{3,12}$

Definition 1.5.
The Fire craker graph denoted by $F_{2,m}$ is obtained the concatenation of 2 stars S_m by linking one leaf from each star.

Theorem 1.7.
The Fire craker $F_{2,m}, m \geq 3$ admits Absolutely Harmonious Labeling.
Proof.
Let $G = F_{2,m}, m \geq 3$.
Let $V(G) = \{v_1, v_2\} \cup \{v_1^1, v_1^2, v_1^3, \dots, v_1^m\} \cup \{v_2^1, v_2^2, v_2^3, \dots, v_2^m\}$
and $E(G) = \{v_1 v_1^i, 1 \leq i \leq m\} \cup \{v_2 v_2^j, 1 \leq j \leq m\} \cup \{v_1^{'}, v_2^m\}$
Here,$G = F_{2,m}, m \geq 3$ is of order $2m + 2$ and size $2m + 1$
Now,Define $f: V(G) \rightarrow \{0,1,2,3, \dots,, q - 1\}$ as follows
$f(v_1) = 0$
$f(v_2) = m + 1$
$f(v_1^i) = i, 1 \leq i \leq m$
$f(v_2^j) = m + j, 1 \leq j \leq m$
Then the induced edge labels are as follows
$f^*(v_1^{'} v_2^m) = a_0$
$f^*(v_2 v_2^i) = a_k;\ 1 \leq i \leq m, 1 \leq k \leq m$
$f^*(v_1 v_1^j) = a_{q-k};\ 1 \leq j \leq m, 1 \leq k \leq m$
It can be easily verifed that $a_0, a_1, a_2, \dots, a_{q-1}$ where $a_i = q - i$ or $q + i, 0 \leq i \leq q - 1$ are the arranged edge labels.
Therefore, f is an absolutely harmonious labeling of the Fire cracker $F_{2,m}, m \geq 3$ and Hence the Fire craker $F_{2,m}, m \geq 3$ is an Absolutely Harmonious Graph.

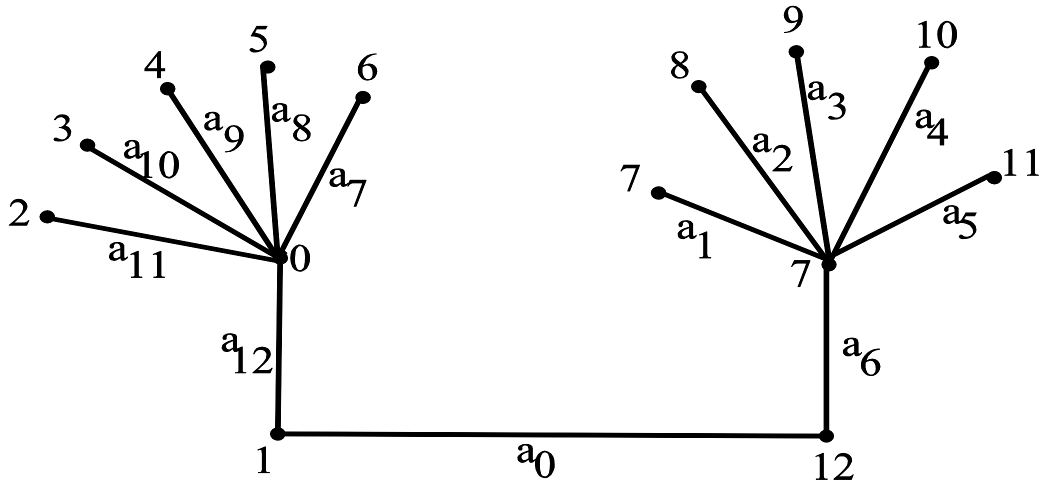

Figure 4: $F_{2,7}$

Theorem 1.8.
$B^2_{n,n}$ is Absolutely harmonious graph.
Proof
Let $G = B^2_{n,n}$
Let $V(G) = \{u, v, u_i, v_i: 1 \leq i \leq n\}, E(G) = \{uv, vv_i, u_iv, v_iu: 1 \leq i \leq n\}$.
Then G is of order $2n + 2$ and size $4n + 1$.
Now,Define $f: V(G) \to \{0,1,2,3, \dots, , q - 1\}$ as follows
Case 1: n=2
$f(u) = 1$
$f(v) = 0$
$f(u_1) = 4$
$f(u_2) = 6$
$f(v_1) = 2$
$f(v_2) = 8$
Case 2:n=3
$f(u) = 1$
$f(v) = 0$
$f(u_1) = 4$
$f(u_2) = 6$
$f(u_3) = 8$
$f(v_1) = 2$
$f(v_2) = 12$
$f(v_3) = 10$
Case 3: $n \geq 4$
$f(u) = 1$
$f(v) = 0$
$f(u_k) = 2i; 3 \leq i \leq n + 2, 1 \leq k \leq n$
$f(v_1) = 2$
$f(v_2) = 4$
$f(v_3) = q - 1$
$f(v_k) = f(v_{k-1}) - 2; 4 \leq k \leq n$
It can be easily verified that f is an absolutely harmonious labeling.
and Hence, $B^2_{n,n}$ is an Absolutely harmonious graph.

Definition 1.6.

A Fan graph is defined as the graph $K_1 + P_n$, where K_1 is the empty graph on one vertex and P_n , $n \geq 2$ is the path graph on n vertices.

Theorem 1.9

The Fan graph F_n is an Absolutely harmonious graph.

Proof.

Let $G = F_n$

Let $V(G) = \{w_0, w_1, w_2, w_3, \cdots, w_n\}$

$E(G) = \{w_0 w_i: 1 \leq i \leq n\} \cup \{w_i w_{i+1}: 1 \leq i \leq n-1\}$

Then G is of order $n+1$ and size $2n-1$

Now,Define $f: V(G) \to \{0,1,2,3,\ldots,,q-1\}$ as follows

$f(w_0) = 1$

$f(w_1) = 0$

$f(w_k) = 2j; 2 \leq k \leq n; 1 \leq j \leq n$

Then the Induced edge labels are arranged as

$f^*(w_0 w_s) = a_{[q-(2r-1)]};\ 1 \leq s \leq n;\ 1 \leq r \leq n$

and the obtained edge labels $a_{2p-1}; 1 \leq p \leq n-1$ can be arranged for the remaining edges $w_i w_{i+1};\ 1 \leq i \leq n-1$.

Hence all the edge labels can be arranged in the above mentioned pattern.

Hence, we observe that $a_0, a_1, a_2, \ldots, a_{q-1}$

where $a_i = q - i$ or $q + i, 0 \leq i \leq q-1$ are the arranged edge labels.

Therefore, f admits absolutely harmonious labeling of the Fan graph.

and Hence,the Fan graph F_n is an Absolutely harmonious graph.

Definition 1.7.

A tree is called a Spider if it has a center vertex c of degree $R \succ 1$ and all the other vertex is either a leaf or with degree 2.Thus a Spider is an amalgamation of k paths with various lengths. If it has x_1's path of length a_1, x_2's path of length a_2,...,.We shall denote the Spider by $SP(a_1^{x_1} a_1^{x_1}, \ldots, a_m^{x_m})$ where $a_1 \prec a_2 \prec \ldots, \prec a_m$ and $x_1 + x_2 + \ldots, + x_m = R$.

Theorem 1.10.

The Spider graph $SP(1^m, 2^t)$ is an Absolutely harmonious graph.

Proof.

Let $G = SP(1^m, 2^t)$

Let $V(G) = \{u, v_i, u_j: 1 \leq i \leq m; 1 \leq j \leq 2t\}$

$E(G) = \{uv_i: 1 \leq i \leq m;\ uu_i: 1 \leq i \leq t;\ u_i u_{t+i}: 1 \leq i \leq t\}$

Then G is of order $n+1$ and size $2n-1$.

Now,Define $f: V(G) \to \{0,1,2,3,\ldots,,q-1\}$ as follows

Case 1: m is odd and t is odd.

$f(u) = 0$

$f(v_i) = i; 1 \leq i \leq m$

$f(u_{2t}) = t$

$$f(u_j) = \begin{cases} m + j \ \ when\ j\ is\ odd\ and\ 1 \leq j \leq t \\ 2m + j\ when\ j\ is\ even\ and\ 1 \leq j \leq t \end{cases}$$

$$f(u_j) = \begin{cases} m + j \ \ when\ j\ is\ even\ and\ t+1 \leq j \leq 2t-1 \\ j \qquad when\ j\ is\ odd\ and\ t+1 \leq j \leq 2t-1 \end{cases}$$

Case 2: m is even and t is even.

$f(u) = 0$

$f(v_i) = 2n + 1; 1 \leq i \leq t; 0 \leq n \leq t-1$

$f(u_1) = m + t$

$f(u_j) = f(u_k) - 2$

$$f(u_j) = \begin{cases} t\,; & j = t+1 \\ 2t+1\,; & j = t+2 \\ (2t+1)+k\,; & t+3 \le j \le 2t; \\ & 1 \le k \le t-2 \end{cases}$$

Case 3:m is odd and t is even.
$f(u) = 0$
$f(v_i) = i; 1 \le \ i \ \le \ m$
$f(u_1) = q - 1$
$f(u_j) = f(u_k) - 2; 2 \le \ j \ \le \ t;\ 1 \le \ k \le \ t - 1$
$f(u_j) = 1; j = t + 1$
$f(u_j) = 2t - 1; j = t + 2$
$f(u_j) = 2t - 3; j = t + 3$
$f(u_j) = 2t - 5; j = t + 4$
$f(u_j) = q - 2k; 1 \le \ k \ \le \ [t/2]; t + 5 \le \ j \le \ 2t$
It can be easily verifed that $a_0, a_1, a_2, \dots, a_{q-1}$ where $a_i = q - i$ or $q + i, 0 \le \ i \le \ q - 1$ are the arranged edge labels in the above three cases .
Therefore, f admits an absolutely harmonious labeling.
and Hence the Spider graph $SP(1^m, 2^t)$ is an Absolutely Harmonious Graph.

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