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Ciric's type common fixed point theorems for six mappings in complete metric spaces

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Abstract
Sessa [13], initiated the tradition of improving commutativity conditions in metrical common fixed point theorems. While doing so Sessa [13] introduced the notion of weak commutativity. Motivated by Sessa [13], Jungck [9] generalized the concept of weak commuting by defining the term compatible mappings and proved that the weakly commuting mappings are compatible but the converse is not true. In recent years, several authors have obtained coincidence point results for various classes of mappings on a complete metric space utilizing these concepts. In this paper, we prove some common fixed point theorems for six mappings involving Ciric's type contractive condition in complete metric spaces. Our work generalizes some earlier results of Cho-Yoo [1] ,Jungck[9], Jang et al. [5],kang and Kim[12] and others. Some examples are also furnished to demonstrate the validity of the hypothesis.

Keywords: Complete metric spaces, fixed points,compatible mapping,weak commutativity.

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1. INTRODUCTION AND PRELIMINARIES:

In recent years several definitions of conditions weaker than commutativity have appeared which facilitated significantly to extend the Jungck's theorem and several others. Foremost among them is perhaps the weak commutativity condition introduced by Sessa [13] which can be described as follows:

1.1 Definition:

Let S and T be mappings of a metric space(X, d) into itself. Then (S, T) is said to be **weakly commuting** pair if$d(STx, TSx) \leq d(Tx, Sx)$for all$x \in X$.

obviously a commuting pair is weakly commuting but its converse need not be true as is evident from the following example.

1.2 Example:

Consider the set $X = [0, 1]$ with the usual metric. Let$Sx = \frac{x}{2}$ and$Tx = \frac{x}{2+x}$ for every$x \in X$. Then for all$x \in X$ $\quad STx = \frac{x}{4+2x}, TSx = \frac{x}{4+x}$

Hence$ST \neq TS$. Thus$S$ and T do not commute.
Again

$$d(STx, TSx) = \left|\frac{x}{4+2x} - \frac{x}{4+x}\right| = \frac{x^2}{(4+x)(4+2x)}$$
$$\leq \frac{x^2}{(4+2x)} = \frac{x}{2} - \frac{x}{2+x} = d(Sx, Tx)$$

and so S and T commute weakly.

Obviously, the class of weakly commuting is wider and includes commuting mappings as subclass.

Jungck [8] has observed that for $X = R$ if $Sx = x^3$ and $Tx = 2x^3$ then S and T are not weakly commuting. Thus it is desirable to a less restrictive concept which he termed as 'compatibility' the class of compatible mappings is still wider and includes weakly commuting mappings as subclass as is evident from the following definition of Jungek [8].

1.3 Definition:

Two self mappings S and T of a metric space(X, d) are **compatible** if and only if$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever$\{x_n\}$ is a sequence in X. such that$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some$t \in X$.

Clearly any weakly commuting pair $\{S,T\}$ in compatible but the converse need not be true as can be seen in the following example.

1.4 Example:

Let $Sx = x^3$ and $Tx = 2x^3$ with $X = R$ with the usual metric. Then S and T are compatible, Since $|Tx - Sx| = |x^3| \to 0$ if and only if $|STx - TSx| = 6|x^9| \to 0$ but $|STx - TSx| \le |Tx - Sx|$ is not true for all $x \in X$, say for example at $x = 1$.

1.5 Proposition:
Let S and T be continuous self mapping on X. Then the pair (S, T) is compatible on X. where as in (Jungck [10], Gajic [2]) demonstrated by suitable examples that if S and T are discontinuous then the two concepts are independent of each other. The following examples also support this observation.

1.6 Example:
Let $X = R$ with the usual metric we define $S, T: X \to X$ as follows.

$$Sx = \begin{cases} 1/x^2 & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1/x^3 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Both S and T are discontinuous at $x = 0$ and for any sequence $\{x_n\}$ in X, we have $d\,(STx_n, TSx_n) = 0$. Hence the pair (S,T) is compatible.

1.7 Example:
Now we define

$$Sx = \begin{cases} 1/x^3, & x > 1 \\ 1, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases} \quad \text{and}$$

$$Tx = \begin{cases} -1/x^3, & x > 1 \\ 1, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases}$$

observe that the restriction of S and T on $(-\infty, 1]$ are equal.
Thus we take a sequence $\{x_n\}$ in $(1, \infty)$. Then $\{Sx_n\} \subset (0,1)$ and $\{Tx_n\} \subset (-1,0)$
Thus, for every n, $TTx_n = 0$, $TSx_n = 1$, $STx_n = 0$, $SSx_n = 1$. So that
$d\,(STx_n, TTx_n) = 0$, $d\,(TSx_n, TTx_n) = 0$ for every $n \in N$.
This shows that the pair (S, T) is compatible of type (A). Now let $x_n = n, n \in N$. Then $Tx_n \to 0$, $Sx_n \to 0$ as $n \to \infty$ and $STx_n = 0$, $TSx_n = 1$ for every $n \in N$ and so $d(STx, TSx) \neq 0$ as $n \to \infty$, hence the pair (S, T) is not compatible.

2. MAIN RESULTS

The following Lemma is the key in proving our result. Its proof is similar to that of Jungck [7].

2.1 Lemma :
Let $\{y_n\}$ be a sequence in a complete metric space (X, d). If there exists a $k \in (0,1)$ such that $d(y_{n+1}, y_n) \le k(y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in X.

Theorem 2.2: Let (X, d) be a complete metric space. Let A, B, S, T, I and J be self mappings from a complete metric space (X, d) into itself satisfying the following conditions:

(i) $AB(X) \subset J(X),\ ST(X) \subset I(X)$ … (1)

(ii) $d(ABx, STy) \le \beta_1 \max\{d(ABx, Ix), d(STy, Jy), \frac{1}{2}[d(ABx, Jy) + d(STy, Ix)], d(Ix, Jy)\} + \beta_2 \max\{d(ABx, Ix), d(STy, Jy)\} + \beta_3 \max\{d(ABx, Jy), d(STy, Ix)\}$ … (2)

for all $x, y \in X$ where $\beta_1, \beta_2, \beta_3 \ge 0$, $0 < \beta = \beta_1 + \beta_2 + 2\beta_3 \le 1$ ($\beta_1, \beta_2, \beta_3$ are non-negative real numbers)
Suppose that
(iii) One of AB, ST, I and J is continuous…(3)
(iv) The pairs (AB, I) and (ST,J) are compatible on X. …(4)
Then the mappings AB, ST, I and J have a unique common fixed point in X.
Furthermore, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J), (T,J) are commuting mappings then the mappings A, B, S, T, I and J have unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point. By (1), since $AB(X) \subset J(X)$, we can choose a point x_1 in X such that $ABx_0 = Jx_1$. Also, since $ST(X) \subset I(X)$, we can fixed a point x_2 with $STx_1 = Ix_2$ and so on. Proceeding in the similar manner, we can define a sequence $\{z_n\}$ in X such that for $n = 0,1,2,3$ ...

$$z_{2n+1} = Jx_{2n+1} = ABx_{2n}, \quad z_{2n} = Ix_{2n} = STx_{2n-1} \quad \ldots (5)$$

Now, we shall show that $\{z_n\}$ is a Cauchy sequence.
Using (2), we have

$$d(z_{2n+1}, z_{2n+2}) = d(ABx_{2n}, STx_{2n+1})$$
$$\le \beta_1 \max\{d(ABx_{2n}, Ix_{2n}), d(STx_{2n+1}, Jx_{2n+1}), \frac{1}{2}[d(ABx_{2n}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n})], d(Ix_{2n}, Jx_{2n+1})\}$$
$$+\beta_2 \max\{d(ABx_{2n}, Ix_{2n}), d(STx_{2n+1}, Jx_{2n+1})\}$$
$$+\beta_3 \max\{d(ABx_{2n}, Jx_{2n+1}), d(STx_{2n+1}, Ix_{2n})\}$$
$$\le \beta_1 \max\{d(z_{2n+1}, z_{2n}), d(z_{2n+2}, z_{2n+1}), \frac{1}{2}[d(z_{2n+1}, z_{2n+1}) + d(z_{2n+2}, z_{2n})], d(z_{2n}, z_{2n+1})\}$$
$$+\beta_2 \max\{d(z_{2n+1}, z_{2n}), d(z_{2n+2}, z_{2n+1})\}$$
$$+\beta_3 \max\{d(z_{2n+1}, z_{2n+1}), d(z_{2n+2}, z_{2n})\}$$

or, $d(z_{2n+1}, z_{2n+2}) \le$

$$\beta_1 \max\{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{1}{2}[d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})], d(z_{2n}, z_{2n+1})\}$$
$$+\beta_2 \max\{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2})\}$$
$$+\beta_3 \max\{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2})\} \ldots (6)$$

where $0 < \beta = \beta_1 + \beta_2 + 2\beta_3 < 1$.

In (6), if $d(z_{2n+1}, z_{2n+2}) > d(z_{2n}, z_{2n+1})$ for some positive integer n, then we get

$$d(z_{2n+1}, z_{2n+2}) \leq \beta\, d(z_{2n+1}, z_{2n+2})$$

which is a contradiction. Then, we obtain

$$d(z_{2n+1}, z_{2n+2}) \leq \beta\, d(z_{2n}, z_{2n+1})$$

Similarly, we get

$$d(z_{2n}, z_{2n+1}) = d(ABx_{2n}, STx_{2n-1})$$

$$\leq \beta_1 \max \{d(ABx_{2n}, Ix_{2n}), d(STx_{2n-1}, Jx_{2n-1}), \frac{1}{2}[d(ABx_{2n}, Jx_{2n-1}) + d(STx_{2n-1}, Ix_{2n})], d(Ix_{2n}, Jx_{2n-1})\} + \beta_2 \max \{d(ABx_{2n}, Ix_{2n}), d(STx_{2n-1}, Jx_{2n-1})\} + \beta_3 \max \{d(ABx_{2n}, Jx_{2n-1}), d(STx_{2n-1}, Ix_{2n})\}$$

$$\leq \beta_1 \max \{d(z_{2n+1}, z_{2n}), d(z_{2n}, z_{2n-1}), \frac{1}{2}[d(z_{2n+1}, z_{2n-1}) + d(z_{2n}, z_{2n})], d(z_{2n}, z_{2n-1})\} + \beta_2 \max \{d(z_{2n+1}, z_{2n}), d(z_{2n}, z_{2n-1})\} + \beta_3 \max \{d(z_{2n+1}, z_{2n-1}), d(z_{2n}, z_{2n})\}$$

$$\leq \beta_1 \max\{d(z_{2n+1}, z_{2n}), d(z_{2n}, z_{2n-1}), \frac{1}{2}[d(z_{2n+1}, z_{2n}) + d(z_{2n}, z_{2n-1})], d(z_{2n}, z_{2n-1})\} + \beta_2 \max\{d(z_{2n+1}, z_{2n}), d(z_{2n}, z_{2n-1})\} + \beta_3 \max \{d(z_{2n+1}, z_{2n}), d(z_{2n}, z_{2n-1})\} \quad (7)$$

In (7), if $d(z_{2n+1}, z_{2n}) > d(z_{2n}, z_{2n-1})$, then we get $d(z_{2n+1}, z_{2n}) \leq \beta d(z_{2n+1}, z_{2n})$, which is a contradiction.

Thus, we get

$$(z_{2n+1}, z_{2n}) \leq \beta d(z_{2n}, z_{2n-1}) \text{ for } n = 1,2,3, \ldots$$

where $0 < \beta < 1$

Now, by induction

$$d(z_{2n}, z_{2n+1}) \leq \beta\, d(z_{2n-1}, z_{2n})$$
$$\vdots$$
$$\vdots$$
$$\leq \beta^n\, d(z_0, z_1)$$

Again, for any $m > n$, we have $(z_n, z_m) \leq d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + \ldots + d(z_{m-1}, z_m) \leq [\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}] d(z_1, z_0) \leq \frac{\beta^n}{1-\beta} d(z_1, z_0)$

This implies that $d(z_n, z_m) \to 0$ as $n, m \to \infty$. Hence, $\{z_n\}$defined by (5) is a Cauchy sequence. Since, X is complete there exists a point z in X such that $\lim_{n\to\infty} z_n = z$i.e.$\{z_n\}$ converges to some $z \in X$.

Therefore, the sequences

$z_{2n+1} = ABx_{2n} = Jx_{2n+1}$ and $z_{2n} = STx_{2n-1} = Ix_{2n}$,

which are subsequences of $\{z_n\}$ also, converges to a point z.

i.e. $\lim_{n\to\infty} ABx_{2n} = \lim_{n\to\infty} Jx_{2n+1} = z$ and $\lim_{n\to\infty} STx_{2n-1} = \lim_{n\to\infty} Ix_{2n} = z$

Now, let I is continuous then the sequences $\{I^2x_{2n}\}$ and $\{IBx_{2n}\}$ converge to the same point Iz. Since $\{AB, I\}$ are compatible on X, so the sequence $\{ABIx_{2n}\}$ also converge to the same point Iz. i.e.

$$I^2x_{2n} \to Iz, ABIx_{2n} \to Iz \text{ as } n \to \infty.$$

By (2), we get

$$d(ABIx_{2n}, STx_{2n-1}) \leq \beta_1 \max \{d(ABIx_{2n}, I^2x_{2n}), d(STx_{2n-1}, Jx_{2n-1}), \frac{1}{2}[d(ABIx_{2n}, Jx_{2n-1}) + d(STx_{2n-1}, I^2x_{2n})], d(I^2x_{2n}, Jx_{2n-1})\} + \beta_2 \max \{d(ABIx_{2n}, I^2x_{2n}), d(STx_{2n-1}, Jx_{2n-1})\} + \beta_3 \max \{d(ABIx_{2n}, Jx_{2n-1}), d(STx_{2n-1}, I^2x_{2n})\}$$

Letting $n \to \infty$, we have

$$d(Iz, z) \leq \beta_1 \max\{d(Iz, Iz), d(z, z), \tfrac{1}{2}[d(Iz, z) + d(Iz, z)], d(Iz, z)\} + \beta_2 \max \{d(Iz, Iz), d(z, z)\} + \beta_3 \max \{d(Iz, z), d(z, Iz)\}$$
$$\leq (\beta_1 + \beta_3)\, d(Iz, z)$$

which is a contradiction as $\beta_1 + \beta_3 < 1$, therefore

$$d(Iz, z) = 0$$

or,$Iz = z$.

Again, by using (2), we get

$$d(ABz, STx_{2n-1}) \leq \beta_1 \max\{d(ABz, Iz), d(STx_{2n-1}, Jx_{2n-1}), \tfrac{1}{2}[d(ABz, Jx_{2n-1}) + d(STx_{2n-1}, Iz)], d(Iz, Jx_{2n-1})\} + \beta_2 \max \{d(ABz, Iz), d(STx_{2n-1}, Jx_{2n-1})\} + \beta_3 \max \{d(ABz, Jx_{2n-1}), d(STx_{2n-1}, Iz)\}$$

on letting $n \to \infty$, we get

$$d(ABz, z) \leq \beta_1 \max\{d(ABz, z), d(z, z), \frac{1}{2}[d(ABz, z) + d(z, z)], d(z, z)\} + \beta_2 \max \{d(ABz, z), d(z, z)\} + \beta_3 \max \{d(ABz, z), d(z, z)\}$$
$$(\beta_1 + \beta_2 + \beta_3)\, d(ABz, z)$$

which is a contradiction as $\beta_1 + \beta_2 + \beta_3 < 1$. Therefore, we get $ABz = z$.

Since $AB(X) \subset J(X)$ and z is in the range of AB i.e. $z \in AB(X)$. Therefore, there exists a point $z' \in X$ such that $z = ABz = Jz'$.

Now,

$$d(z, STz') = d(ABz, STz')$$

$$\leq \beta_1 \max\{d(ABz, Iz), d(STz', Jz'), \tfrac{1}{2}[d(ABz, Jz') + d(STz', Iz)], d(Iz, Jz')\} + \beta_2 \max \{d(ABz, Iz), d(STz', Jz')\} + \beta_3 \max \{d(ABz, Jz'), d(STz', Iz)\}$$
$$\leq \beta_1 \max\{d(z, z), d(STz', z), \frac{1}{2}[d(z, z) + d(STz', z)], d(z, z)\} + \beta_2 \max \{d(z, z), d(STz', z)\} + \beta_3 \max \{d(z, z), d(STz', z)\}$$

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or, $d(z, STz') \leq (\beta_1 + \beta_2 + \beta_3)\, d(STz', z)$ which is a contradiction as $\beta_1 + \beta_2 + \beta_3 < 1$, which gives $z = STz'$. Therefore, $z = STz' = Jz'$, which shows that z' is a coincidence point of ST and J.

Since, ST and J are compatible on X and $Jz' = STz' = z$. Therefore, we have $d(JSTz', STJz') = 0$.
Hence, $Jz = JSTz' = STJz' = STz$
or, $Jz = STz$.
By (2), we get

$$d(z, Jz) = d(ABz, STz) \leq \beta_1 \max\{d(ABz, Iz), d(STz, Jz), \tfrac{1}{2}[d(ABz, Jz) + d(STz, Iz)], d(Iz, Jz)\} + \beta_2 \max\{d(ABz, Iz), d(STz, Jz)\} + \beta_3 \max\{d(ABz, Jz), d(STz, Iz)\}$$

$$\leq \beta_1 \max\{d(z, z), d(Jz, Jz), \tfrac{1}{2}[d(z, Jz) + d(z, Jz)], d(z, Jz)\} + \beta_2 \max\{d(z, z), d(Jz, Jz)\} + \beta_3 \max\{d(z, Jz), d(Jz, z)\}$$

or, $d(z, Jz) \leq (\beta_1 + \beta_3)\, d(z, Jz)$ a contradiction so that $z = Jz = STz$, which shows that z is a common fixed point of AB, ST, I and J.

Now, suppose that AB is continuous so that the sequences $\{AB^2x_{2n}\}$ and $\{ABIx_{2n}\}$ converge to ABz. Since, $\{AB, I\}$ are compatible on X, it follows that $\{IABx_{2n}\}$ also converges to ABz i.e. $AB^2x_{2n} \to ABz, IABx_{2n} \to ABz$ as $n \to \infty$.
By (2), we have

$$d(AB^2x_{2n}, STx_{2n-1}) \leq \beta_1 \max\{d(AB^2x_{2n}, IABx_{2n}), d(STx_{2n-1}, Jx_{2n-1}), \tfrac{1}{2}[d(AB^2x_{2n}, Jx_{2n-1}) + d(STx_{2n-1}, IABx_{2n})], d(IABx_{2n}, Jx_{2n-1})\} + \beta_2 \max\{d(AB^2x_{2n}, IABx_{2n}), d(STx_{2n-1}, Jx_{2n-1})\} + \beta_3 \max\{d(AB^2x_{2n}, Jx_{2n-1}), d(STx_{2n-1}, IABx_{2n})\}$$

which on letting $n \to \infty$, reduces to

$$d(ABz, z) \leq \beta_1 \max\{d(ABz, ABz), d(z, z), \tfrac{1}{2}[d(ABz, z) + d(ABz, z)], d(ABz, z)\} + \beta_2 \max\{d(ABz, ABz), d(z, z)\} + \beta_3 \max\{d(ABz, z), d(z, ABz)\} \leq (\beta_1 + \beta_3) d(ABz, z)$$

which is a contradiction, yielding thereby
$ABz = z$ as $\beta_1 + \beta_3 < 1$.

Since z is in the range of AB and $AB(X) \subset J(X)$, there always exists a point z' such that $Jz' = z = ABz$. Then

$$d(AB^2x_{2n}, STz') \leq \beta_1 \max\{d(AB^2x_{2n}, IABx_{2n}), d(STz', Jz'), \tfrac{1}{2}[d(AB^2x_{2n}, Jz') + d(STz', IABx_{2n})], d(IABx_{2n}, Jz')\} + \beta_2 \max\{d(AB^2x_{2n}, IABx_{2n}), d(STz', Jz')\} + \beta_3 \max\{d(AB^2x_{2n}, Jz'), d(STz', IABx_{2n})\}$$

which on letting $n \to \infty$ reduces to

$$d(z, STz') \leq \beta_1 \max\{d(z, z), d(STz', z), \tfrac{1}{2}[d(z, z) + d(STz', z)], d(z, z)\} + \beta_2 \max\{d(z, z), d(STz', z)\} + \beta_3 \max\{d(z, z), d(STz', z)\} \leq (\beta_1 + \beta_2 + \beta_3) d(STz', z),$$

a contradiction which yields
$z = STz' = Jz'$.

Thus, the pair (ST, J) has a coincidence point z'.
Since, the pair (ST, J) is compatible on X and $Jz' = STz' = z$, we have $d(JSTz', STJz') = 0$ [by def.of compatible].
Hence, $Jz = J(STz') = ST(Jz') = STz$, which shows that $STz = Jz$.
Further, by (2), we have

$$d(ABx_{2n}, STz) \leq \beta_1 \max\{d(ABx_{2n}, Ix_{2n}), d(STz, Jz), \tfrac{1}{2}[d(ABx_{2n}, Jz) + d(STz, Ix_{2n})], d(Ix_{2n}, Jz)\} + \beta_2 \max\{d(ABx_{2n}, Ix_{2n}), d(STz, Jz)\} + \beta_3 \max\{d(ABx_{2n}, Jz), d(STz, Ix_{2n})\}$$

which on letting $n \to \infty$, we get

$$d(z, STz) \leq \beta_1 \max\{d(z, z), d(STz, STz), \tfrac{1}{2}[d(z, STz) + d(STz, z)], d(z, STz)\} + \beta_2 \max\{d(z, z), d(STz, STz)\} + \beta_3 \max\{d(z, STz), d(STz, z)\} \leq (\beta_1 + \beta_3)\, d(z, STz),$$

a contradiction which implies that $STz = z = Jz$ as $\beta_1 + \beta_3 < 1$
Since, $ST(X) \subset I(X)$ and $STz = z$, then there exists a point z'' in X such that $Iz'' = z$. Thus,

$$d(ABz'', z) = d(ABz'', STz) \leq \beta_1 \max\{d(ABz'', Iz''), d(STz, Jz), \tfrac{1}{2}[d(ABz'', Jz) + d(STz, Iz'')], d(Iz'', Jz)\} + \beta_2 \max\{d(ABz'', Iz''), d(STz, Jz)\} + \beta_3 \max\{d(ABz'', Jz), d(STz, Iz'')\}$$

$$\leq \beta_1 \max\{d(ABz'', z), d(z, z), \tfrac{1}{2}[d(ABz'', z) + d(z, z)], d(z, z)\} + \beta_2 \max\{d(ABz'', z), d(z, z)\} + \beta_3 \max\{d(ABz'', z), d(z, z)\}$$

or, $d(ABz'', z) \leq (\beta_1 + \beta_2 + \beta_3)\, d(ABz'', z)$,

a contradiction which implies that
$ABz'' = z$ as $\beta_1 + \beta_2 + \beta_3 < 1$.

Again, since (AB, I) are compatible on X and $ABz'' = Iz'' = z$, we have
$d(IABz'', ABIz'') = 0$.

Therefore, $Iz = I(ABz'') = AB(Iz'') = ABz$.
Hence, $ABz = Iz = z$. Thus, we have proved that z is a common fixed point of AB, ST, I and J.
Instead of AB or I, if the mappings ST or J is continuous, then the proof that z is a common fixed point of AB, ST, I and J is similar.

To show that z is unique, let u be another fixed point of AB, ST, I and J. Then

$d(z,u) = d(ABz, STu)$
$\leq \beta_1 \max\{d(ABz,Iz), d(STu,Ju), \frac{1}{2}[d(ABz,Ju) + d(STu,Iz)], d(Iz,Ju)\}$
$+\beta_2 \max\{d(ABz,Iz), d(STu,Ju)\} + \beta_3 \max\{d(ABz,Ju), d(STu,Iz)\}$
$\leq \beta_1 \max\{d(z,z), d(u,u), \frac{1}{2}[d(z,u) + d(u,z)], d(z,u)\}$
$+\beta_2 \max\{d(z,z), d(u,u)\} + \beta_3 \max\{d(z,u), d(u,z)\}$
or, $d(z,u) \leq (\beta_1 + \beta_3)\, d(z,u)$,

a contradiction yielding thereby
$z = u$ as $\beta_1 + \beta_3 < 1$.

Finally, we will prove that z is also a common fixed point of A, B, S, T, I and J. Let both the pairs (AB,I) and (ST,J) have a unique common fixed point z. Then
Az = A(ABz) = A(BAz) = AB(Az)
Az = A(Iz) = I(Az)
Bz = B(ABz) = B(A(Bz)) = BA(Bz)= AB(Bz)
Bz = B(Iz) = I(Bz)
which implies that (AB,I) has common fixed points which are Az and Bz. We get, thereby Az = z = Bz = Jz = ABz, by virtue of uniqueness of common fixed point of pair (AB, I).
Similarly, using the commutativity of (S,T), (S,J) and (T, J), Sz = z = Tz = Jz = STz can be shown.
Now, we claim that Az = Sz (Bz = Tz), a common fixed point of both the pairs (AB,I) and (ST,J). We have

d(Az, Sz) = d(A(ABz), S(STz))
= d(A(BAz), S(TSz))
= d(AB(Az), ST(Sz))

$\leq \beta_1 \max\{d(AB(Az),I(Az)), d(ST(Sz),J(Sz)), \frac{1}{2}[d(AB(Az),J(Sz)) + d(ST(Sz),J(Az))], d(I(Az),J(Sz))\}$
$+\beta_2 \max\{d(AB(Az),I(Az)), d(ST(Sz),J(Sz))\}$
$+\beta_3 \max\{d(AB(Az),J(Sz)), d(ST(Sz),I(Az))\}$

$\leq \beta_1 \max\{d(Az,Az), d(Sz,Sz), \frac{1}{2}[d(Az,Sz) + d(Sz,Az)], d(Az,Sz)\}$
$+\beta_2 \max\{d(Az,Az), d(Sz,Sz))\} + \beta_3 \max\{d(Az,Sz), d(Sz,Az)\}$
or, $d(Az,Sz) \leq (\beta_1 + \beta_3)\, d(Az,Sz)$,

a contradiction which implies that $Az = Sz$.
Similarly, Bz = Tz can be shown. Thus, z is the unique common fixed point of A, B, S, T, I and J.
The following corollary follows immediately from our theorem (2.2).

Corollary 2.3: Let (X,d) be a complete metric space. Let A, B, S, T, I and J be self mappings from a complete metric space (X,d) into itself satisfying the following conditions:

(i) $AB(X) \subset J(X),\ ST(X) \subset I(X)$
(ii) $d(ABx,STy) \leq \beta_1 \max\{d(ABx,Ix), d(STy,Jy), \frac{1}{2}d(ABx,Jy), \frac{1}{2}d(STy,Ix), d(Ix,Jy)\} + \beta_2 \max\{d(ABx,Ix), d(STy,Jy)\} + \beta_3 \max\{d(ABx,Jy), d(STy,Ix)\}$

for all $x, y \in X$ where $\beta_1, \beta_2, \beta_3 \geq 0, \beta_1 + \beta_2 + 2\beta_3 \leq 1$ ($\beta_1, \beta_2, \beta_3$ are non-negative real numbers)
Suppose that
(iii) One of AB, ST, I and J is continuous.
(iv) The pairs (AB, I) and (ST,J) are compatible on X.

Then the mappings AB, ST, I and J have a unique common fixed point in X.
Furthermore, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J), (T,J) are commuting mappings then the mappings A, B, S, T, I and J have unique common fixed point.
If we put AB = A, ST = B in theorem (2.2), we get the following, which generalize the result of Jang et al. [5] in metric spaces.

Corollary 2.4: Let (X,d) be a complete metric space. Let A, B, S and T be self mappings from a complete metric space (X,d) into itself satisfying the conditions:

(i) $A(X) \subset T(X), B(X) \subset S(X)$
(ii) $d(Ax,By) \leq \beta_1 \max\{d(Ax,Sx), d(By,Ty), \frac{1}{2}[d(Ax,Ty) + d(By,Sx)], d(Sx,Ty)\}$
$+\beta_2 \max\{d(Ax,Sx), d(By,Ty)\} + \beta_3 \max\{d(Ax,Ty), d(By,Sx)\}$

For all $x, y \in X$, where $0 < \beta = \beta_1 + \beta_2 + 2\beta_3 < 1$ ($\beta_1, \beta_2, \beta_3$)are non-negative real numbers).
(iii) Suppose that one of A, B, S and T is continuous.
(iv) The pairs (A,S) and (B,T) are compatible on X then A, B, S and T have a unique common fixed point in X.

Taking AB=A, ST=B, $\beta_2 = 0$ in theorem (2.2), we obtain the following, which generalize the result of Cho-Yoo [1] in metric spaces.

Corollary 2.5: Let (X,d) be a complete metric space. Let A, B, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions:

(i) $A(X) \subset T(X), B(X) \subset S(X)$
(ii) $d(Ax,By) \leq \beta_1 \max\{d(Ax,Sx), d(By,Ty), \frac{1}{2}[d(Ax,Ty) + d(By,Sx)], d(Sx,Ty)\}$

$+\beta_3 \max\{d(Ax,Ty), d(By,Sx)\}$

For all $x, y \in$X, where $0 < \beta = \beta_1 + 2\beta_3 < 1$ (β_1, β_3 are non-negative real numbers).

(iii) Suppose that one of A, B, S and T is continuous.

(iv) The pairs $(A, S)$and (B, T) are compatible on X then A, B, S and T have a unique common fixed point in X.

If we put AB=A, ST=B, $\beta_2 = 0$ and $\beta_3 = 0$ in theorem (2.2), we obtain the following, which improve and generalize the result of Kang-Kim [12] and Jungck [9] in metric spaces.

Corollary 2.6: Let (X, d) be a complete metric space. Let A, B, S and T be mappings from a complete metric space (X, d)into itself satisfying the conditions:

(i) $A(X) \subset T(X), B(X) \subset S(X)$

(ii) $d(Ax, By) \leq$
$\beta_1 \max\{d(Ax,Sx), d(By,Ty), \frac{1}{2}[d(Ax,Ty) + d(By,Sx)], d(Sx,Ty)\}$

For all $x, y \in$X, where $0 < \beta_1 < 1$ (β_1 is non-negative real number).

(iii) Suppose that one of A, B, S and T is continuous.

(iv) The pairs $(A, S)$and (B, T) are compatible on X then A, B, S and T have a unique common fixed point in X.

Example 2.7: Consider $X = [0,1]$ with the usual metric defined by

$d(x, y) = \|x - y\| = |x - y|$ and $F = R =$ Real Banach space.

Define self mappings A, B, S, T, I and J by

$Ax = \frac{2x}{3}$, $Bx = \frac{3x}{4}$, $Sx = \frac{x}{4}$, $Tx = \frac{4x}{5}$, $Ix = \frac{x}{4}$ and $Jx = \frac{3x}{4}$ for all $x \in X$, respectively.

Then, all the hypothesis oftheorem (2.2) are satisfied for

$\beta_1 = \frac{1}{5}$, $\beta_2 = \frac{1}{3}$ and $\beta_3 = \frac{1}{20}$ where $\beta_1 + \beta_2 + 2\beta_3 < 1$. Hence, 0 is a unique common fixed point of A, B, S, T, I and J.

Now, we give some examples to illustrate our corollary (2.4).

Example 2.8: Let $X = [0, \infty)$ with the usual metric defined by $d(x,y) = \|x-y\| = |x - y|$ and $E = R =$ Real Benach space.

Define self mappings A, B, S, T : $X \to X$ by

$Ax = Bx = \frac{1}{8}x + 1$, $Sx = Tx = \frac{1}{2}x + 1$for all $x \in X$, respectively.

Now, we get

$d(Ax, Ay) = \frac{1}{4} d(Sx, Sy)$
$\leq \frac{1}{4} max\{d(Ax,Sx), d(Ay,Sy), \frac{1}{2}[d(Ax,Sy) + d(Ay,Sx)], d(Sx,Sy)\}$
$+\beta_2 max\{d(Ax,Sx), d(Ay,Sy)\} + \beta_3 max\{d(Ax,Sy), d(Ay,Sx)\}$

For all $x, y \in X$, where $0 \leq \beta_2 + 2\beta_3 < \frac{3}{4}$. Here, all the conditions of the corollary (2.4) are satisfied except the condition of compatibility of the pair (A, S). Therefore, A and S don't have a common fixed point in X.

Example 2.9: Let $X = [0,1]$ with the usual metric defined by

$d(x,y) = \|x-y\| = |x - y|$and $E = R =$ Real Benach space.

Define A, B, S and T: $X \to X$ by

$$Ax = 0, Bx = \begin{cases} \frac{1}{4}\ if\ x = \frac{1}{2} \\ \frac{1}{4}x\ if\ x \neq \frac{1}{2} \end{cases}, Sx = x,\ Tx = \begin{cases} 1\ if\ x = \frac{1}{2} \\ x\ if\ x \neq \frac{1}{2} \end{cases}$$

for all $x \in X$ respectively. We get

$$d(Ax, By) = \begin{cases} \frac{1}{4} = \frac{1}{3} d(By,Ty)\ if\ y = \frac{1}{2} \\ \frac{1}{4}y = \frac{1}{3} d(By,Ty)\ if\ y \neq \frac{1}{2} \end{cases}$$

$\leq \frac{1}{3} max\{d(Ax,Sx), d(By,Ty), \frac{1}{2}[d(Ax,Ty) + d(By,Sx)], d(Sx,Ty)\}$
$+q\ max\{d(Ax,Sx), d(By,Ty)\} + r\ max\{d(Ax,Ty), d(By,Sx)\}$

For all $x, y \in X$, where $0 \leq \beta_2 + 2\beta_3 < \frac{2}{3}$.

Therefore, all the conditions of corollary (2.4) are satisfied. Consequently, 0 is a unique common fixed point of A, B, S and T.

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