EXISTENCE RESULTS OF CONFORMABLE FRACTIONAL INTEGRO DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN BANACH SPACES

[1]D.Mallika
[1]Department of Mathematics,
Hindusthan College of Arts and Science,Coimbatore-641028,
Tamil Nadu, India.
Email:malliabilash@gmail.com
[2]S.Suganya
[2]Department of Science and Humanities,
Arjun College of Technology,Coimbatore-20,
Tamil Nadu, India.
[3]C.Prabha
[3]Department of Mechanical Engineering,
Arjun College of Technology,Coimbatore-20,
Tamil Nadu, India.
[4]A.Rohini
[4]Department of Science and Humanities,
KGISL Institute of Technology,Coimbatore-35,
Tamil Nadu, India.

Abstract
This paper explores the existence of solutions for the conformable integro differential equations of fractional order with infinite delay in Banach spaces by making use of Contraction mapping, the fixed point techniques Leray Schaudernonlinear alternative and Schafer fixed point theorem.
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1.Introduction

Liouville and Riemann revealed the fundamental representation of the fractional derivative at the end of the nineteenth century. The idea of a non-integer derivative and integral, as opposed to the conventional integer order differential and integral, was first introduced in 1695. In the same year, Leibniz was questioned by L'Hopital on a specific notation for the nth-derivative of the linear function f(x) = x, $d^n/(dx^n)$ when n = 1/2. Study of L'Hopital's and Leibniz's fractional calculus was initially primarily restricted to the best mathematicians.Few mathematicians likeFourier, Euler, and Laplacehave experimented with fractional calculus and its mathematical ramifications.Their own notation and method was revealed for defining the concept of a fractional order integral or derivative. Two of the most well-known definitions in the area of fractional calculus are the Riemann-Liouville definition and the Grunwald-Letnikov definition. The mathematical theory necessary for studying fractional calculus was created at the turn of the 20th century.
They offer a potent tool to link the memory of various substances and the environment of the heritage. A power law memory kernel of the nonlocal interactions can be used to produce generalised and non-integer order differential equations in time and space known as fractional order differential equations. Many physical phenomena, including nonlinear seismic oscillations, control systems, elasticity, electric drives, circuits systems, continuum mechanics, heat transport, quantum physics, fluid mechanics, and signal analysis, make use of FDE.For basics of fractional systems one can make reference to the books [3,9,15] and papers[1,2,7,8,13,14] and the references cited therein.

In the study of qualitative as well as quantitative theory for functional differential equations, phase space is a fundamental idea. Hale and Kato introduced the seminormed space obeying appropriate axioms in chapter four [4]. (see also Benchorra et.al [17] and Y.Zhou et.al [18]). We advise the reader to read Hino et albook .'s [5] for a comprehensive discussion on this subject

R.Khali.et.al [10] has introduced the conformable fractional differential operator and the rapid development of conformable fractional differential equations has been made see [11,12,21,22,23],and the references therein.

In this paper we are concerned with the existence of mild solutions of CFID with infinite delay of the form

$u(\mathbb{t})=\psi(\mathbb{t})$; $\mathbb{t}\epsilon$ $(-\infty, a]$ (1.1)-(1.2)

$T_{a^+}{}^{r}$ $[u(\mathbb{t})-k(\mathbb{t},s,u_s)]= g(\mathbb{t}, u_s, \int_a^t E(\mathbb{t},s,u_s)\, ds$

where ψ: $(-\infty, a] \rightarrow X$, $k, g: J \times B \times B \rightarrow X$ are given continuous functions and B is called a phase space that will be specified later and J= [a,b]

2.Preliminaries:

The Banach space of all real continuous functions on X with the norm is C(J,X).

$$\|x\|_\infty = \sup_{t\epsilon I}|x(t)|$$

By $L^1(X)$ we denote the Banach space of measurable functions x:J→R with are Lebesgue integrable, equipped with the norm.

$$\|x\|L^1 = \int_0^T x(t)dt$$

For the purposes of this work, we assume that the Phase space (B, ||. ||B) is aseminormed linear space of functions mapping (−∞, a] into X, and that it satisfies the following fundamental axiomsintroduced by Hale and Kato [4]

(A-1)If u :(−∞, a]→R, and u_0∈ B, then for every $\mathbb{t}$∈ [0, a] the following conditions hold:

(i) $u_{\mathbb{t}}$is in B,

(ii) $\|u_t\|_B \le K(t) \sup\|u(s)\|: 0 \le s \le t + P(\mathbb{t})\|u_0\|_B$

(iii) $\|u(\mathbb{t})\| \le H\|u_t\|_B$,

where H > 0 is a constant, K : [0, b]→ [0,∞) is continuous, P: [-∞,a) → [-∞,a) is locally bounded andH,K,M are independent of u(·).

(A-2) For the function u(·) in (A-1), $u_{\mathbb{t}}$is a B-valued continuous function on [0, b].

(A-3) The space B is complete.

Definition:2.1

The conformable fractional derivative of order 0<r≤1 starting from a of the function x:J→R is defined by,

$T^r_{a+}\, x(t)=\frac{u\left(t+h(t-a)^{1-r}\right)-u(t)}{h}$

Particularly, if u is differentiable, then

$T^r_{a+}\, x(t) = (t-a)^{(1-r)}\frac{d}{dt}u(t)$

Definition:2.2 (Conformable fractional integral)

The conformable fractional integrals of order r> 0 of a function x:J→R is defined by,

$I^r_{a+}\, x(t) =\int_0^t (s-a)^{(r-1)}x(s)ds, \quad \mathbb{t}\, \epsilon\, J$

Example:2.1

For 0 <r ≤ 1 and λ ∈R, we have

$T_0{}^{r}\, \lambda =0,\ T_0{}^{r}\mathbb{t}^{r}= \lambda\mathbb{t}^{(\lambda-r)}$

$T_0{}^{r}e^{\lambda\mathbb{t}} =\lambda\mathbb{t}^{1-r}e^{\lambda\mathbb{t}}\ ,\ \ \mathbb{t}\, \epsilon J$

Lemma:2.1

Let 1≤ r>0,and u∈ C(J), then $T^r_{a+} I^r_{a+}\, x(\mathbb{t})=x(\mathbb{t})$. Further, if x is differentiable on J, then $T^r_{a+} I^r_{a+}\, x(\mathbb{t})=x(\mathbb{t})-x(a)$.

Definition:2.3

A mapping $\mathcal{F}:\mathbb{X} \to \mathbb{X}$ is said to be contraction if there exists a real number k, 0≤k<1, such that $\|\mathcal{F}(u) - \mathcal{F}(v)\| \leq k\|u - v\|$ for all u,v∈ $\mathbb{X}$. Note that $\|.\|$ indicates a norm in $\mathbb{X}$.

Definition: 2.4 Banach fixed point theorem

If $\mathbb{X}$ is a Banach space and $\mathcal{F}:\mathbb{X} \to \mathbb{X}$ is a Contraction mapping then F has a unique fixed point.

Definition: 2.5 Krasnoselskii's fixed point theorem

Letχ_1,χ_2 be two operators satisfying,

a) $\chi_1$is contraction and
b) $\chi_2$is completely continuous.

Then either,

i)The operator equation χ_1x+χ_2x=x has a solution or

ii)The set $\eta = \{u\epsilon x: \mu\chi_1\left(\frac{x}{\lambda}\right) + \mu\chi_2\, u = u\}$ is bounded for $\mu \in (0,1)$.

We will now present the idea of mild solution u ϵ B of (1.1)-(1.2).This equation is equivalent to the subsequent integral equation

$$u(t) = \begin{cases} \psi(t)\ ;\ t\epsilon\ (-\infty, 0] \\ \psi(a)-k(a, u_a, a)+k(t, u_t, \int_0^t E((t,s,u_s)ds)+\int_a^t (s-a)^{\alpha-1} g(t,u_s,\int_0^t E(t,s,u_s)ds);\ t\epsilon\ [0,a] \end{cases}$$

Let x(·) : (−∞, a]→R be the function defined by

$$x(t) = \begin{cases} 0, \text{ if } t\in [0, a], \\ \psi(t), \text{ if } t\in (-\infty, 0]. \end{cases}$$

Then $x_0 = \psi$. For each yϵC([0, a],X) with y(0) = 0, we denote by $\bar{y}$the function defined by

$$\bar{y}(t) = \begin{cases} y(t), \text{ if } t \in [0, a], \\ 0, \text{ if } t\in (-\infty, 0] \end{cases}$$

We can decompose u(·) as u(t) = y(t) + x(t), 0 ≤ t ≤a, which implies = y_t+ x_t, for every 0 ≤ t ≤a .Set C_0={u∈ C([0,a],X): u_0 = 0} and let $\|.\|_b$be the seminorm in C_0 defined by$\|u\|_b = \|u_0\|_B + Sup\{|u(t)| : 0 \leq t \leq a\} = Sup\{|u(t)|: 0 \leq t \leq a\}$ for u ϵ C_0

3. Existence Results:

The existence of solutions for the structure (1.1)–(1.2) under various fixed point theorems is presented and demonstrated in this section.

Now we list the assumptions that come next.

Hypothesis:

H1) k:J×B×B→X is continuous and we can find functions L_1,L_2 ∈ C[J,R^+] in a way that,
$\|k((t,\xi,x) - k((t,\zeta,y\| \leq L_1\|\xi - \zeta\|_B + L_2\|x - y\|_B$ and
$\|k((t,\xi,t)\| \leq L_k\|\xi\|_B$

H2) g:J×B×B→X is continuous and we can find functions L_3,L_4 ∈ C[J,R^+] in a way that,
$\|g((t,\xi,x) - g((t,\zeta,y\| \leq L_3\|\xi - \zeta\|_B + L_4\|x - y\|_B$

H3) E:J×B×B→is continuous and we can find functions L_eT∈ C[J,R^+] in a way that,
$\|E((t,s,x) - E(t,s,y\| \leq L_eT\|x - y\|_B$

H4) The function Φ(t):J→ R+ is determined by,
$\Phi((t) = K[(L_1+L_2TL_e)+(L_3+L_4TL_e)]$ t ∈ J,0<Φ(t)<1

Theorem:3.1

Assume that the conditions (H1)-(H4) hold and u ϵ X.Then the problem(1.1)-(1.2) has a unique mild solution .

The following estimates are based on the aforementioned hypothesis.

$$\|k((t, u_t + x_t, \int_0^t E(t,s,u_s + x_s)ds) - k(t,\overline{u_t} + x_t, \int_0^t E(t,s,\overline{u_s} + x_s)ds)\|_B$$
$$\leq L_1\|u_t - \overline{u_t}\| + L_2\|\int_0^t E(t,s,u_s + x_s) - E(t,s,\overline{u_s} + x_s)ds\|_B$$

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$\leq L_1\|u_t-\overline{u_t}\|+L_2TL_e\|u_{\mathfrak{t}}-\bar{u}_{\mathfrak{t}}\|_B$

$\leq (L_1+L_2TL_e)\|u_{\mathfrak{t}}-\bar{u}_{\mathfrak{t}}\|_B$

$\left\|\int_a^t (s-a)^{\alpha-1}\left[g\left(\mathfrak{t},u_s+x_s,\int_0^t E(s,\tau,u_\tau+x_\tau)d\tau\right)-g\left(\mathfrak{t},\overline{u_s}+x_s,\int_0^t E(s,\tau,\overline{u_\tau}+x_\tau)d\tau\right)\right]ds\right\| \leq \int_a^t (s-a)^{\alpha-1}\,ds\left\|\begin{bmatrix} g\left(\mathfrak{t},u_s+x_s,\int_0^t E(s,\tau,u_\tau+x_\tau)d\tau\right)\\ -g\left(\mathfrak{t},\overline{u_s}+x_s,\int_0^t E(s,\tau,\overline{u_\tau}+x_\tau)d\tau\right)\end{bmatrix}\right\|$

$\leq \int_a^t (s-a)^{\alpha-1}ds\ [L_1\|u_s-\overline{u_s}\|_X+L_2TL_e)\|u_\tau-\bar{u}_\tau\|_B]$

$\leq \int_a^t (s-a)^{\alpha-1}ds\ [(L_1+L_2TL_e)\|u_\tau-\bar{u}_\tau\|_B]$

Therefore, $\|Zu(\mathfrak{t})-Z\overline{u(\mathfrak{t})}\| \leq [(L_1+L_2TL_e)+(M_3+L_4TL_e)\int_a^t (s-a)^{\alpha-1}\,ds]\|u_{\mathfrak{t}}-\bar{u}_{\mathfrak{t}}\|_B$

$\leq [(L_1+L_2TL_e)+(M_3+L_4TL_e)\int_a^t (s-a)^{\alpha-1}\,ds]K\sup_{s\epsilon[0,t]}\left\|u(\mathfrak{t})-\overline{u(\mathfrak{t})}\right\|_B$

$\leq \Phi(t)\left\|u(\mathfrak{t})-\overline{u(\mathfrak{t})}\right\|_B$

We can deduce that contains a single fixed point u, which is a mild solution of the model (1.1)-(1.2) on $(-\infty\ a]$from the presumption (H4) and from the perspective of the contraction mapping principle.

We continue to examine the existence result for the models (1.1)-(1.2).using the Leray-Schauder nonlinear alternative fixed point theorem.

Lemma 3.1[20,p.135].Let $\mathbb{C}$ be a Banach space , $\mathbb{E}$ a closed ,convex subset of $\mathbb{C}$,$\mathfrak{U}$ an open Subset of $\mathbb{E}$ and $0\ \epsilon\ \mathfrak{U}$. Suppose that $\mathcal{F}$: $\overline{\mathfrak{U}}\to\mathbb{E}$ is a continuous, compact (that is, $\mathcal{F}$ ($\overline{\mathfrak{U}}$) is a relatively compact subset of $\mathbb{C}$) map. Then either

i) $\mathcal{F}$has a fixed point in $\overline{\mathfrak{U}}$, or

ii) there is a u $\epsilon\ \partial\mathfrak{U}$ (the boundary of $\mathfrak{U}$ in $\mathbb{E}$) and $\lambda\ \epsilon(0,1)$ with u=$\lambda\,\mathcal{F}(u)$.

(H1*) The function k: J×B×B→X is completely continuous and there exists constants $M_k>0$ and$\widehat{M_k}>0$ such that $\|k(\mathfrak{t},\varphi,\zeta)\|_X\leq M_k(\|\varphi\|_B+\|\zeta\|_B)+\widehat{M_k}$ for all $(\mathfrak{t},\varphi,\zeta)$ is measurable

(H2*) i)There exists a continuous increasing function φ:[-∞, a) →[-∞, a) and a continuous function p_1:X→[-∞, a) such that $\|g(\mathfrak{t},\zeta,\zeta_1\|_B\leq p_1(\mathfrak{t})\,\psi(\|\zeta\|_B+\|\zeta_1\|_B)$ for all $(\mathfrak{t},\zeta,\zeta_1)\epsilon J\times B\times B$

ii) There exists a constant $\mathcal{N}>0$ such that

$$\frac{\mathcal{N}}{\frac{\xi^*C_1}{\mu}+\frac{\xi^*t^\alpha}{\mu\ \alpha}\|p_1\|_\infty\psi(\xi^*\|w_1\|)\ ds}>1$$

whereμ =1- K $M_k(1+L_eT)$ and $C_1=\|\psi(a)\|+KL_k\|u_a\|+\widehat{M_k}$

Theorem:3.2

Suppose that the assumptions (H1) through (H4) and (H1*) through (H2*) are true. The model (1,1)-(1,2) thus has a singular mild solution on (-∞ , a]

Proof:

Now we develop the subsequent hypothesis

(H5) g: J× $B\times B\to$ X is continuous and there exists functions L_g such that $\|g(t,0,0)\|\leq L_g$

Transform the problem(1.1)-(1.2)into a fixed point problem.Let usrecognize the operator $Z: B\to B$ characterized by,

$$(Zu)(\mathfrak{t})=\begin{cases}\psi(\mathfrak{t})\ \mathfrak{t}\in(-\infty,\mathrm{a}]\\ \psi(a)-k(a,u_a,a)+k\left(\mathfrak{t},u_t,\int_0^t E(\mathfrak{t},s,u_s)ds\right)\\ +\int_a^t (s-a)^{\alpha-1}g(\mathfrak{t},u_s,\int_0^t e(s,\tau,u_\tau)d\tau)ds\ \mathfrak{t}\in(-\infty,a]\end{cases}$$

To prove the operator $\overline{Z}$ is completely continuous, we split the operator $\overline{Z}: B\to B$.

Presently for $\mathfrak{t}\ \epsilon$J we split $\overline{Z}$ as$\overline{Z}_1+\overline{Z}_2$where,

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$$\overline{(Z_1)}(t) = \psi(a) - k(a, u_a, a) + k\left(t, u_t + x_t, \int_0^t E(t, s, u_s)ds\right)$$

$$\overline{(Z_2)}(t) = \int_a^t (s-a)^{\alpha-1} g(t, u_s, \int_0^t e(s, \tau, u_\tau)d\tau)ds \quad t \in (-\infty, a]$$

Step:1

$\overline{Z}B_r \subset B_r$ for some r>0. We assert the existence of a positive integer r such that $\overline{Z}B_r \subset B_r$. If it is not true, then a function for each positive number r ,$x^r(.) \in B_r$.

But $Z(x^r) \notin B_r$..

$\|(Zx^r)(t)\|$>r for some $t \in J$, we sustain,

r$\leq \|(Zx^r(t)\|$

$\leq \|\psi(a)\| + \|k(a, u_a, a)\| + \|k(t, u_t, \int_0^t E(t, s, u_s)ds\|$

$+ \|\int_a^t (s-a)^{\alpha-1} g(t, u_s, \int_0^t E(s, \tau, u_\tau)d\tau)ds\|$

$\leq I_1 + I_2 + I_3 \rightarrow$ (3.1)

$I_1 = \|\psi(a)\|$

$I_2 = \|k(a, u_a, a)\|$

$\leq L_k\|u_a\|$

$$\leq K L_k r$$

$I_3 = \|k(t, u_t, \int_0^t E(t, s, u_s)ds\|$

$$= \left\|k(t, u_t + x_t, \int_0^t E(t, s, u_s + x_s)ds\,) - k(t, 0,0)\right\| + \|k(t, 0,0)\|$$

$\leq L_1\|u_t\| + L_2\|\int_0^t E(t, s, u_s + x_s)ds\|$

$\leq L_1\|u_t\| + L_2 L_e T\|u_t\|$

$\leq K(L_1 + L_2 L_e T)r$

$I_4 = \|\int_a^t (s-a)^{\alpha-1} g(t, u_s, \int_0^t E(s, \tau, u_\tau)d\tau)ds\|$

$\leq \int_a^t (s-a)^{\alpha-1} \|g(t, u_s + x_s, \int_0^t E(s, \tau, u_\tau + x_\tau)d\tau\|ds$

$\leq L_1\|u_t\| + L_2\|\int_0^t E(t, \tau, u_\tau + x_\tau)d\tau\| \int_a^t (s-a)^{r-1}$ ds

$\leq \frac{t^\alpha}{\alpha} K(L_3 + L_4 L_e T)r$

Therefore (3.1) becomes

$\|(Z x^r)(t)\| \leq \|\psi(a)\| + K L_k r + (L_1 + L_2 L_e T)r + \int_a^t (s-a)^{r-1}$ ds$(L_3 + L_4 L_e T)r$

$$\leq r$$

Step:2

To prove $\overline{Z_1}$ is a contraction.

$\|\overline{Z_1}u(t) - \overline{Z_1}\overline{u(t)}\| \leq \|k(t, u_t + x_t, \int_0^t E(t, s, u_s + x_s)ds) - k(t, \overline{u_t} + x_t, \int_0^t E(t, s, \overline{u_s} + x_s)ds)\|_B$

$\leq L_1\|u_t - \overline{u_t}\| + L_2\|\int_0^t E(t, s, u_s + x_s) - E(t, s, \overline{u_s} + x_s)ds\|_B$

$\leq L_1\|u_t - \overline{u_t}\| + L_2 T L_e\|u_t - \overline{u_t}\|_B$

$\leq (L_1 + L_2 T L_e)\|u_t - \overline{u_t}\|_B$

$\leq \Lambda\, \|u(t) - \overline{u(t)}\|_B$ [Since $K(L_1 + L_2 T L_e) = \Lambda$]

Step:3

We shall prove that $\overline{Z_2}$ maps bounded sets into bounded sets.

$\|\overline{Z_2}u(t)\| \leq \|\int_a^t (s-a)^{\alpha-1} g t, u_s + x_s, \int_0^t E(s, \tau, u_\tau + x_\tau)d\tau)ds\|$

$\leq \int_a^t (s-a)^{\alpha-1} \|g(t, u_s + x_s, \int_0^t E(s, \tau, u_\tau + x_\tau)d\tau\|ds$

$\leq L_3\|u_t\| + L_4\|\int_0^t E(t, \tau, u_\tau + x_\tau)d\tau\| \int_a^t (s-a)^{r-1}$ ds

$\leq \int_a^t (s-a)^{\alpha-1}$ds $K(L_3 + L_4 L_e T)r + L_g$

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$\leq$ r

To prove $\overline{Y_2}$ is equicontinuous

Let $\varepsilon > 0$ be small, $0 < \mathfrak{t}_1 < \mathfrak{t}_2 \leq T$. For any x ϵ B_r ={x ϵ B: $\|x\| \leq r\}$, and $\mathfrak{t} \,\epsilon (-\infty, a]$

$$\|\overline{Z_2}x(\mathfrak{t}_2) - \overline{Z_2}x(\mathfrak{t}_1)\| \leq \int_0^{t_2} \left\|(\mathfrak{t}_2 - a)^{\alpha-1}\, ds\, g(\mathfrak{t}, u_s + x_s, \int_0^s E(s,\tau,u_\tau + x_\tau)d\tau)\right\|$$
$$+\int_0^{t_1} \left\|(\mathfrak{t}_1 - a)^{\alpha-1}\, ds\, g(\mathfrak{t}, u_s + x_s, \int_0^s E(s,\tau,u_\tau + x_\tau)d\tau)\right\|$$
$$\leq \int_0^{t_1} \left\|(\mathfrak{t}_2 - a)^{\alpha-1}\, ds\, g(\mathfrak{t}, u_s + x_s, \int_0^s E(s,\tau,u_\tau + x_\tau)d\tau)\right\|$$
$$+\int_{t_1}^{t_2} \|(\mathfrak{t}_2 - a)^{\alpha-1}\, ds\, g(\mathfrak{t}, u_s + x_s, \int_0^s E(s,\tau,u_\tau + x_\tau)d\tau)\|$$
$$-\int_0^{t_1} \|(\mathfrak{t}_1 - a)^{\alpha-1}\, ds\, g(\mathfrak{t}, u_s + x_s, \int_0^s E(s,\tau,u_\tau + x_\tau)d\tau)\|$$
$$\leq \int_0^{t_1} [(\mathfrak{t}_2 - a)^{\alpha-1} - (\mathfrak{t}_1 - a)^{\alpha-1}]ds\| g(\mathfrak{t}, u_s + x_s, \int_0^s E(s,\tau,u_\tau + x_\tau)d\tau)\|$$
$$+\int_{t_1}^{t_2} (\mathfrak{t}_2 - a)^{\alpha-1}\, ds\| g(\mathfrak{t}, u_s + x_s, \int_0^s E(s,\tau,u_\tau + x_\tau)d\tau)\|$$

$\to$ 0 as $\mathfrak{t}_2 \to \mathfrak{t}_1$.

which intimates that $\overline{Z_2}$ (.) is equicontinuous . Therefore by Arzela Ascoli theorem , we realize that the operator $\overline{Z_2}$ is completely continuous.

Step:4

We demonstrate the existence of an open set $\mathfrak{U} \subset C(I,X)$ with u $\notin \lambda Z(u)$ for $\mu \epsilon (0,1)$ and u $\epsilon\, \partial\mathfrak{U}$.

Let $\mu \epsilon (0,1)$ and

$$u(\mathfrak{t}) = \lambda \begin{Bmatrix} \psi(a) - k(a, u_a, a) + k\left(\mathfrak{t}, u_t, \int_0^t E(\mathfrak{t}, s, u_s + x_s)ds\right) \\ + \int_a^t (s-a)^{\alpha-1} g(\mathfrak{t}, u_s, \int_0^t e(s,\tau,u_\tau + x_\tau)d\tau)ds) \end{Bmatrix}, \quad \mathfrak{t} \epsilon\, J.$$

$$\|u(\mathfrak{t})\| \leq \|\psi(a)\| + \|k(a,u_a,a)\| + \left\|k\left(\mathfrak{t}, u_t + x_t, \int_0^t E(\mathfrak{t}, s, u_s + x_s)ds\right)\right\|$$
$$+\left\|\int_a^t (s-a)^{\alpha-1} g(\mathfrak{t}, u_s + x_s, \int_0^t e(s,\tau,u_\tau + x_\tau)d\tau)ds\right\|$$

$$\leq \|\psi(a)\| + KL_k\|u_a\| + KM_k(1+L_eT)\|u(s)\| + \widehat{M_k} + \int_a^t (s-a)^{\alpha-1}\ p_1(\mathfrak{t})\psi\,\{\|u_s\| + L_eT\|u_\mathfrak{t}\|\}$$

$\|u((\mathfrak{t})\| \leq \frac{C_1}{\mu} + \frac{1}{\mu}\int_a^t (s-a)^{\alpha-1} p_1((\mathfrak{t})\ \psi(\xi^*\|u(s)\|)$ ds where $\xi^* = K(1+L_eT)$

Then $\xi^*\|u(\mathfrak{t})\| \leq \frac{\xi^* C_1}{\mu} + \frac{\xi^*}{\mu}\int_a^t (s-a)^{\alpha-1} p_1(\mathfrak{t})\ \psi(\xi^*\|u(s)\|)$ ds

We contemplate the function specified by $w_1(\mathfrak{t}) = \sup\{\xi^*\|u(s)\|: 0 \leq s \leq \mathfrak{t}\}$, $\mathfrak{t}\,\epsilon\, J$

We have $w(\mathfrak{t}) \leq \frac{\xi^* C_1}{\mu} + \frac{\xi^*}{\mu}\int_a^t (s-a)^{\alpha-1} p_1(\mathfrak{t})\ \psi(\xi^*\|w_1(s)\|)$ ds

Then $\|w_1\| \leq \frac{\xi^* C_1}{\mu} + \frac{\xi^*}{\mu}\frac{t^\alpha}{\alpha}\|p_1\|_\infty \psi(\xi^*\|w_1(s)\|$

And consequently we have $\dfrac{\|w_1\|}{\frac{\xi^* C_1}{\mu} + \frac{\xi^*}{\mu}\frac{t^\alpha}{\alpha}\|p_1\|_\infty \psi(\xi^*\|w_1\|)\ ds} \leq 1$

Then by (H2*)(ii) , there exists $\mathcal{N}$ such that $\|w_1\| \neq \mathcal{N}$. We shall define

$\mathfrak{U} = \{u\ \epsilon\ C(J,X): \|u\| < \mathcal{N}\}$

From the option of $\mathfrak{U}$, there is no u $\epsilon\, \partial\mathfrak{U}$ u=μ Z(u) for some $\mu\ \epsilon$ (0,1). As a result of lemma 3.1, we infer that the operator has a fixed point u which is a solution of the given problem on (-∞,a].

Theorem 3.3

Let $\mathbb{X}$ be a Banach space, and $\mathfrak{D}: \mathbb{X} \to \mathbb{X}$ be a completely continuous operator. Then either

i) $\mathfrak{D}$ has a fixed point or

ii) The set Ω = {x $\epsilon\mathfrak{U}$: x= $\mu\mathfrak{D}$ (x) ,0< $\lambda < 1$ } is unbounded.

Theorem 3.4

Assume that the hypotheses (H1)-(H4) and (H1*) is satisfied. Moreover, assume(H3*)

There exists functions p,q ϵ C(I,X) such that

$\|g(\mathbb{t},\mu,\nu)\| \leq$ p($\mathbb{t}$)+q($\mathbb{t}$) ($\|\mu\|_B$+$\|\nu\|_B$) , $\mathbb{t}\epsilon$ J, $\mu,\nu\ \epsilon$ B×B.

Then the problem (1.1)-(1.2) has atleast one mild solution on (-∞,a] provided that

{1-K $M_k(1 + LeT)$- $K\ \frac{\mathbb{t}^{\alpha}}{\alpha}\|q\|_{\infty}\psi(\xi^*)$}<1 .

Proof:

The operator Z is defined as in theorem 3.1.As in theorem 3.2 we can prove that the operator Z is completely continuous. Now , we prove that the set

Ω = {uϵB : x= λ Z(u),0< $\mu < 1$ } is bounded.

Let u $\epsilon\Omega$ be any element.Then as in the theorem 3.2, we have each $\mathbb{t}\epsilon\ J$,

$\|u(\mathbb{t})\| \leq C_1$ +K $M_k(1+L_eT)$ $\|u(s)\|$+$\int_a^t (s-a)^{\alpha-1}\|p\|_{\infty}ds$

$+\int_a^t (s-a)^{\alpha-1} K\|q\|_{\infty}\,\psi(\xi^*\|u(s)\|)$ ds

where C_1=$\|\psi(a)\|$+K$L_k\|u_a\|$+$\widehat{M_k}$ and consequently we have

$\|u\|_T \leq C_1 + \frac{T^{\alpha}}{\alpha}\|p\|_{\infty}ds\ \{1-\mu_1{}^{-1}\}$

whereμ_1 ={1-K $M_k(1 + LeT)$- $K\ \frac{\mathbb{t}^{\alpha}}{\alpha}\|q\|_{\infty}\psi(\xi^*)$}.

The set is consequently bounded. Z must have at least one fixed point, according to Theorem 3.3. Since the operator must also have a fixed point, the mild solution of (1.1)- (1.2)

Conclusion

The existence results for conformable fractional integro differential equations (CFIDE) with infinite delay in Banach spaces have been investigated in this paper using the Banach contraction principle, the Leray-Shauder nonlinear alternative, and the Schafer fixed point theorem. The need to broaden the research to demonstrate more qualitative and quantitative characteristics including stability, controllability, and other attributes is critical given how significant the concept is in illuminating real-world happenings.

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