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REVIEW OF CHANGE OF VARIABLE THEOREM FOR THE RIEMANN INTEGRAL

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ABSTRACT
In this paper we has reviewed about this, Kestelman,H he was first proved the Riemann integral theorem ofchange in variablefor its modern in 1961 . theadditional his result of theoremwith the inverse statement was by preiss in 1970. for that they give another proofs of that theorems we are reviewing within thethat formulations.(here f is a function and bounded with its domain) we are except on using concept of Riemann integral of the change of variable.

Keywords: Real Analysis; Riemann Integral; Change of Variable

INTRODUCTION

In this paper, we denote $[x_1, x_2]$ as the closed interval connecting the points$x_1, x_2 \in R$, and denote $R[x_1, x_2]$ as the classificationof all Riemannintegral of real valued functions on$[x_1, x_2]$.

Theorem 1.1.(Kestelman, H) letconsider the$g \in R[x_1, x_2], A \in R,$ and f

$$G(y) = \int_a^t g(x)dx + A……(1.1)$$

and $f \in R(G[x_1, x_2])$. for this, $(f \circ G)g \in R[x_1, x_2]$ and the followsthat changeing of variable formulae holds that:

$$\int_{G(x_1)}^{G(x_2)} f(t)dt = \int_{x_1}^{x_2} f\,[G(y)]g(y)dy \;……(1.2)$$

In this theorem(see [1.2]) fundamental this result with the following statement.

Theorem 1.2. Let us considerg $\in R[x_1, x_2]$, G is defined by (1.1), f is bounded on $[x_1, x_2] = G[x_1, x_2]$ and $(f \circ G)g \in R[x_1, x_2]$.Then $f \in R[x_1, x_2] \subset R[G(x_1), G(x_2)]$ and in the change ofvariable Formulaof the equation (1.2) holds that.

For this , in a number ofpapers (see [3–6]), theProof of this Theorems 1.1 were given within the same as formation. For Thisnote isrequired of bounded of the function f on $[x_1, x_2] := G([x_1, x_2])$ in Theorem1.2. At this same time, the condition for the function f ofbounded on$[G(x_1), G(x_2)]$ it is needed for the function of existence of the condition ofboundedness of integral both side.

Theorem 1.3. Let us considerg $\in R[x_1, x_2]$, G is defined by (1.1), f is bounded on $I := [G(x_1), G(x_2)]$ and $(f \circ G)g \in R[x_1, x_2]$. Then, $f \in R(I)$ and the change of variable Formula (1.2)holds.

since the proof of Theorem 1.3, we has need the following lemma.

Lemma 1. If $g, gh \in R[x_1, x_2]$, then $g|h| \in R[x_1, x_2]$.

Proof. By Lebesgue's criterion, let be functions$g$ and gh are both continuous on$[x_1, x_2]$. Let $y \in [x_1, x_2]$ be the point wise continuous. If $h$is continuous at y_0, then$g|h|$ is continuous at y_0. If h is discontinuous at y_0, then the equality $g(y_0) = 0$ must hold because otherwise, h must be continuous at y_0 as a quotient of continuous functions gh and g. Then,
we have the follow that

$g(y)h(y) \rightarrow g(y_0)h(y_0) = 0.$

As will as

$$\therefore g(y)|h(y)| = g(y)h(t)\text{sgn}h(y) \rightarrow 0 = g(y_0)|h(y_0)|$$

as$y \rightarrow y_0$, which means the continuity of the function$g|h|$ at y_0, and thus, its continuity a.e.on$[x_1, x_2]$. Thus, $g|h| \in R[x_1, x_2]$ by Lebesgue's criterion.

Proof of Theorem 1.3.By inthis theorem, there we needed$M_1 > 0$such that
$|f(t)| \leq M_1$for all $x \in I$. For all $n \in N$, let $C_n := M_1 + n$ and define

$$\forall\, t \in [x_1, x_2] := \leq G([x_1, x_2])$$

Then the belowfunction:

$$f_n(t) = \begin{cases} f(t), \text{if } |f(t)| \leq Cn; \\ Cn, \text{if } f(t) > Cn \quad ; \\ -Cn, \text{if } f(t) < -Cn; \end{cases}$$

From the given definition for all $n \in N$, we obtain the bounded offn as well as the following equality:

$$f_n|_I = f|_I \ldots\ldots(1.3)$$

for every$n \in N$ for all $x \in [x_1, x_2]$, we obtain that:

$$|fn(t)| \leq |(t)|, \ldots\ldots(1.4)$$

and $\forall\, x \in [x_1, x_2]$, we have the following:

$$fn(t) \rightarrow f(t) \ldots\ldots(1.5)$$

as$n \rightarrow \infty$. Next, we show that $(fn \circ G)g \in R[x_1, x_2]$ for all $n \in N$. For each $n \in N$, we have the following

For thisformula:

$$f_n = \min\left\{\max\{f, -C_n\}, C_n\right\}$$
$$= \frac{1}{14}(\mathrm{f} - \mathrm{C_n} - |\mathrm{f} - \mathrm{C_n}| + |3\mathrm{C_n} + \mathrm{f} - |\mathrm{f} - \mathrm{C_n}||),$$

from which, for $h := f \circ G$, we obtain the follows below equality:

$$(fn \circ G)g = \frac{1}{14}(h - C_n - |h - C_n| + |3\mathrm{C_n} + h - |h - C_n||)g \ldots\ldots (1.6)$$

Since by the concept of the theorem $g, gh \in R[\mathrm{x_1}, \mathrm{x_2}]$, then by Lemma 1, we have $g|h - C_n| \in R[x_1, x_2]$, and thus,$g|3\mathrm{C_n} + h - |h - C_n|| \in R[\mathrm{x_1}, \mathrm{x_2}]$ by the same lemma. Finally, (1.6) implies that $(fn \circ G)g \in R[\mathrm{a}, \mathrm{b}]\ \forall\, n \in N$

Since the$(f \circ G)g$ is theintegral andbounded function,$\exists M_2 > 0$
such that for all $n \in N$, $\mathrm{x} \in [\mathrm{x_1}, \mathrm{x_2}]$ holds the that:

$$|fnG(\mathrm{y})\ g(\mathrm{y})| \le |fG(\mathrm{y})\ g(\mathrm{y})| \le M_2,$$

For this Additionally, for ally $\in [\alpha, \beta]$ as $n \to \infty$, wefollows that:

$$fnG(\mathrm{y})\ g(\mathrm{y}) \to f\ G(\mathrm{y})\ g(\mathrm{y}).$$

By (1.3), using Theorem 1.2 and theconvergence theorem of the Riemann integral ofArzela's bounded (see [7]), as n → ∞ we obtains:

$$\int_{G(x_1)}^{G(x_2)} f(t)dt = \int_{G(x_1)}^{G(x_2)} f_n(t)dt = \int_{x_1}^{x_2} f_n(G(y))g(y)dy \to \int_{x_1}^{x_2} f(G(y))g(y)dy,$$

Which complete review of the verification of (2) and the proof of the theorem.

Theorem 1.4: By **[Sarkhle D N]** If TheRiemann integral change of variables formulae (1.1) is valid if either

(1) The Riemann integralof g in the range ofF, or

(2) The bounded function g andthe rangeF and its (1.1) exists the Riemann integral ofthe righthand side of

the proof this theoremit follows

Lemma 2 If z is a Lipschitz function on $[\mathrm{x_1}, \mathrm{x_2}]$ and$y' \le 0$ almost everywhere then
$\mathrm{y}(\mathrm{x_2}) \le \mathrm{y}(\mathrm{x_1})$. If y is Lipschitz and$y' = 0$ almost everywhere then $\mathrm{y}(\mathrm{x_2}) = \mathrm{y}(\mathrm{x_1})$.

Proof. Form here we wasprove the lemma of first part. Let M> 0 is anLipschitz constant ofx.
and
Given the ε > 0, ∃ a countableof disjoint open intervalsI_m, m = 1, 2, . . . , n coversthe set where y' either negative or is positive and such that

$$\sum_{1}^{\infty}|I_n| < \frac{\in}{2M}.$$

however

$$y' < \frac{\in}{2(x_2 - x_1)}$$

On

$$[x_1, x_2] \setminus \bigcup_{1}^{n} I_n,$$

And the LUBfor all t ∈ $[x_1, x_2]$ such that

$$y(t) - y(t) \leq \frac{\in (t - x_1)}{2(x_2 - x_1)} + \sum_{1}^{n} M|I_m \cap (x_1, t)|$$

consequentlyy$y(x_2) - y(x_1) \leq \varepsilon$.

Proof ofTheorem1.4.

Let G = {y ∈ (y_1, y_2); Fis the continuous ony}, G= {y∈G ; the functionf(y) = 0}, G_-= {y∈G; E(y) <0},andG_+ = {y∈G; E(t) > 0}.

(1) letus assume

$$G(y) = \int_{F(y_1)}^{F(y)} f,$$

$$E(y) = \int_{\underline{y_1}}^{y} g \circ F f.$$

Such thatG we haveG(x) = E' (x) = 0andthe bounded function of g, $F'(y) = 0$ and the continuousfunction e on y and F(y) = 0.we assume thaty ∈ G_+, and the positive integerhs t F > 0 on [y − h, y + h].we know |h| > x_1 then

$$G(y + h) - G(y) = F(y + h) - F(x)$$

S t

$$(G - f)'(y) = 0 \ldots\ldots(1.7)$$

fory$\in F_+$. Similarly for y$\in F_-$. Consequently (1.6)constrainsin the functionGand such that .on $[\alpha, \beta]$. using second lemma having$\mathrm{G} - \mathrm{E} = \mathrm{Z}$

$$\int_{F(y_1)}^{F(y_2)} g = \int_{\underline{y_1}}^{y_2} g \circ Ff \ldots\ldots(1.8)$$

Similarly

$$\int_{F(y_1)}^{F(y_2)} g = \int_{y_1}^{\overline{y_2}} g \circ F f \ldots\ldots(1.9)$$

$$\int_{\underline{y_1}}^{y_2} g \circ F f = \int_{y_1}^{\overline{y_2}} g \circ F f = \int_{F(y_1)}^{F(y_2)} g$$

(2).we Assume

$$\mathrm{E(x)} = \underline{\int_{\mathrm{F(y_1)}}^{\mathrm{F(y_2)}} \mathrm{g},}$$

$$\mathrm{G(x)} = \int_{y_1}^{y} g \circ F f.$$

Herethe consider thethe bounded function fwithin (2) appearthat vile but, in fact, is needed. If$\mathrm{g}(1/\mathrm{m}) = \mathrm{m}$ and $\mathrm{f}(1/\mathrm{m}) = 0$ for $\mathrm{m} = 1, 2, \ldots,$ and $\mathrm{g(y)} = 0, \mathrm{f(y)} = 1$ if $y_1 = 0$ &$y_2 = 1$ And integral exists on one hand side of (1) but does not exists on the other hand of (1) .

2Riemann integral through primitives:

Definition :By [4] consider the functionf : $[\mathrm{x}_1, \mathrm{x}_2] \rightarrow \mathrm{R}$ be an bounded function.
And thegeneralized primitive ofthe function f,E : $[x_1, x_2] \rightarrow \mathrm{R}$ st for all$\mathrm{t}_1, t_2 \in [\mathrm{x}_1, \mathrm{x}_2], t_1 < t_2$, we have

$$\inf_{\mathrm{t}_1 \le \mathrm{t} \le \mathrm{t}_2} \mathrm{f(t)} \le \frac{\mathrm{F(t_2)} - \mathrm{F(t_1)}}{\mathrm{t_2} - \mathrm{t_1}} \le \sup_{t_1 \le t \le t_2} \mathrm{f(t)} \ldots\ldots(2.1)$$

Iffor EveryGeneralized primitives is not necessary to be differentiable, yetquicklyalike the relations between of the function of f its derivative.

Theorem 2.1.suppose that the function $\mathrm{f}:[\mathrm{x}_1,\mathrm{x}_2]\rightarrow\mathrm{R}$ be bounded andgeneralized primitive of f is F $\forall t_1\in[\mathrm{x}_1,\mathrm{x}_2)$for this

$$\min\left\{f(t_1),\lim_{t_1\to t_2^+} inf\ f(t_2)\right\}\leq D_+f(t_1)$$
$$\leq \mathrm{D}_+\mathrm{f}(\mathrm{t}_1)\leq \max\left\{\mathrm{f}(\mathrm{t}_1),\lim_{\mathrm{t}_2\to\mathrm{t}_1^+}\sup\mathrm{f}(\mathrm{t}_2)\right\}\ldots\ldots(2.2)$$

Now
$\forall\, t_1\in(\mathrm{t}_1,\mathrm{t}_2]$for this

$$\min\{f(t_1),\lim_{t_2\to t_1^+} inf(t_2)\}\leq D_-f(t_1)$$
$$\leq \mathrm{D}^-\mathrm{f}(\mathrm{t}_1)\leq \max\{\mathrm{f}(\mathrm{t}_1),\lim_{\mathrm{t}_2\to\mathrm{t}_1^+}\sup\mathrm{f}(\mathrm{t}_1)\}\ldots\ldots(2.3)$$

S t, The continuous function f on $\mathrm{t}\in[\mathrm{x}_1,\mathrm{x}_2]$ then $\mathrm{F}'(t_1)=\mathrm{f}(t_1)$

Proof. Here they are proved only one side–hand inequality only in (2.2) and the
Other handinequalities can be provedbysimilarmethod. We assume that$t_1\in[\mathrm{x}_1,\mathrm{x}_2)$ tobe fixed; they take limit of inferior in theinequality oneof (2.1) and see below the better result:

$$D_+f(t_2)=\lim_{t_2\to t_1^+} inf\ \frac{F(t_2)-f(t_1)}{t_2-t_1}\geq\lim_{t_2\to t_1^+} inf\ \inf_{t_1\leq t\leq t_2} f(t)=\lim_{t_2\to t_1^+}\inf_{t_1\leq t\leq t_1} f(t)$$
$$=\lim_{t_2\to t_1^+}\min\left\{f(t_1),\inf_{t_1\leq t\leq t_2} f(t)\right\}$$
$$=\min\left\{f(t_1),\lim_{t_2\to t_1^+} inf\ f(t_1)\right\}$$

For this when the f is continuous is an immediate and their derivative exist therefore of (2.2) and (2.3).

Theorem2.2 Here we was assume that f is continuous functionin the $[x_1,x_2]$ on mappingf $:[\mathrm{x}_1,\mathrm{x}_2]\rightarrow$ Rthen every generalized primitive of f is a primitive in, conversely, primitives andusual sense are generalized primitives.
Proof. By above theoremwe ensures that generalized primitives satisfy the $\mathrm{F}'=\mathrm{f}$ in $[\mathrm{x}_1,\mathrm{x}_2]$, so they are primitives in the usual sense.

Against ,If F is a primitive of f then$\forall\mathrm{t}_1,\mathrm{t}_2\in[\mathrm{x}_1,\mathrm{x}_2],t_1<t_2$,by the existence theorem ofMean Valuecontracttheof some z ∈ (x, y) st

$$\frac{F(t_2)-F(t_1)}{t_2-t_1}=F'(x)=f(x),$$

Imply(2.1).

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Theyare in a establishthe position of our first test for the Riemann integral in terms of generalized primitives.using The Barrow's Rule for computing integrals of the generalized primitives it isincluded in the test.

Theorem 2.3:we assume that g is bounded functionand $C \in R$ mappingg: $[x_1, x_2] \to R$and Thebelow given two statements are equivalent:

1. If thefunction g isintegral on $[x_1, x_2]$ and $\int_{x_1}^{x_2} g(t)dt = c$;

2. The g satisfies $G(x_2) - G(x_1) = C$.

For the Proof.we assume that G: $[x_1, x_2] \to R$ be a arbitrary generalized primitive of g and We Consider thisI $= \{t_1, t_2, t_3, \ldots.. t_n\}$ be a sub intervalsof $[x_1, x_2]$.
using (2.1), then

$$F(x_2) - F(x_1) = \sum_{i=1}^{n} [G(t_i) - G(t_{i-1})] \leq U(f, I),$$

similarly, $F(x_2) - F(x_1) \geq L(f, I)$. However I is casualand g is integral functionin the$[x_1, x_2]$,here they finishthe$G(x_2) - G(x_1) = \int_{x_1}^{x_2} g(t)dt = C$.
vice versa, they assumed to different accountsG_* and G^* , is indefinationof generalized primitives of g. forthe condition of theorem 2.2 verify that

$$\underline{\int_{x_1}^{x_2}} g(x)dx = G_*(x_2) - G_*(x_1) = C = G^*(x_2) - G^*(x_1) = \overline{\int_{x_1}^{x_2}} G(x)dx.$$

Here finely reviewed the change of variable inRiemann integral.

3. Some applications

Example 1

The following example illustrates Theorem 1.3 in use: let a:= 1,b:=2,f(y) := 2y,
F(y) := y^2 and

$$g(t) = \begin{cases} \frac{1}{\sqrt{t}}, t < 0 \\ 0, t = 0 \end{cases}$$

Clearly, f
is unbounded on G([-3, 1]) = [0, 5], but there exists

$$\int_{1}^{5} \frac{dt}{\sqrt{t}} = \int_{G(a)}^{G(b)} g(t)dt = \int_{a}^{b} g(F(y))f(y)dy = \int_{-3}^{1} 2sgn(y)dy = 2$$

For this some other applications of our result, we obtain as a resultthe change of a variable of Riemann improper integral of the theorem on (in one direction) under general conditions.

Corollary 1(by1.3)**.** *we assume that* $x_1 < x_2$, $y_1 < y_2$, *f is bounded on* $[x_1, x_3]$ *for all* $x_3 \in (x_1, x_2)$, $f \in R[y_1, y_3]$ *for all* $y_3 \in (y_1, y_2)$,

$$F(t) = \int_{y_1}^{y} f(t)dx = I$$

And

$$\lim_{t \to t-} \int_{t_1}^{t} g(F(t))dt = I$$

Then the following holds

$$\lim_{s \to \beta -} \int_{\alpha}^{s} f(t)dt = I$$

Conclusion

Finely My conclusion with the framework is that the about change of variable integral defined by Sarkhel and Torchinsky, A may be aforce to be reckoned with. the Riemann integral in undergraduate studies gives the scholar more powerful tools and asan example , letting the differentiability of internal function which is common in first-year real analysis be theoreticallywell-grounded. Some problems with the integral are alluded to within the preceding discussion and thus I don't think it'll be possible, albeit it had been through with the simplest of intentions, to the Riemann theory entirely. Especially if wewould like a neater thanks to generalize to many changes of a variable. Since Kesselman's theorem is far simpler todetermine for the Riemann integral—and to more general spaces. We do have a characterization of the change ofvariable integral by Riemann, however, which could be an honest way for college kids to become familiarized with theconcept.

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